GROUPOIDS, ROOT SYSTEMS AND WEAK ORDER I

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ABSTRACT. This is the first of a series of papers concerned with certain structures called rootoids and protorootoids, the definition of which is abstracted from formal properties of Coxeter groups with their root systems and weak orders. A protorootoid is defined to be a groupoid (i.e. a small category with every morphism invertible) equipped with a representation in the category of Boolean rings and a corresponding 1-cocycle. Weak preorders of protorootoids are preorders on the sets of their morphisms with fixed codomain, defined by the natural ordering of the corresponding values of the cocycle in the Boolean ring in which they lie. A rootoid is a protorootoid satisfying axioms which imply that its weak preorders are partial orders embeddable as order ideals of complete ortholattices. Rootoids and protorootoids may be studied alternatively as groupoids with abstract root systems. The most novel results of these papers involve categories of rootoids with favorable properties including existence of small limits and of functor rootoids, which provide interesting new structures even for finite Weyl groups such as the symmetric groups. This first introductory paper gives basic definitions and terminology, and explains the correspondence between cocycles and abstract root systems. Examples discussed here include rootoids attached to Coxeter groups, to real simplicial hyperplane arrangements and to orthogonal groups. Rudimentary properties of subclasses of rootoids called principal rootoids and complete rootoids are established. Principal (respectively, complete principal) rootoids naturally generalize arbitrary (respectively finite) Coxeter systems.

INTRODUCTION

Introduction to this work. Coxeter groups are an important class of discrete groups which occur naturally in many areas of mathematics. A Coxeter group W has a presentation as a group generated by a set S of involutory simple generators subject to braid relations, which specify the order of the product of each pair of simple generators. The presentation is determined by a square matrix, the Coxeter matrix, with its entries in $\mathbb{N}_{\geq 1} \cup \{\infty\}$ equal to the orders of these products. In applications related to Lie theory, W often occurs in conjunction with root systems, which are combinatorial structures on which W acts. Classically, root systems Φ are subsets of real vector spaces which encode a description of an action of W as a finite or discrete real reflection group. More abstract notions of root systems are also useful, for example in the theory of buildings. Examples of Coxeter groups which are obviously important are the finite and affine Weyl groups, and especially the symmetric groups S_n, which arise as the Weyl groups of the special linear groups and Lie algebras.

A root system Φ of a Coxeter group W may be viewed abstractly as a W-set (the elements of which are called roots) equipped with a commuting free action by the

group $\{\pm 1\}$ and a distinguished set Φ_+ of $\{\pm 1\}$-orbit representatives, called the set of positive roots. There is a bijection $\alpha \mapsto s_\alpha$ from the set of positive roots Φ_+ to the set $T = \{\, wsw^{-1} \mid w \in W, s \in S \,\}$ of reflections. For any $w \in W$, define $\Phi_w := \Phi_+ \cap w(-\Phi_+)$ and $N(w) := \{\, s_\alpha \mid \alpha \in \Phi_w \,\}$. Let $\wp(T)$ be the Boolean ring of subsets of T. As a function $N\colon W \to \wp(T)$, N is called the reflection cocycle of W (see [10]). It is well known that $N(w) = \{\, t \in T \mid l(tw) < l(w) \,\}$ where $l(w) = \min(\{\, n \in \mathbb{N} \mid w = s_1 \cdots s_n, s_i \in S \,\})$ is the standard length of w.

In the case of S_n, its simple generators are the adjacent transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$, the reflections are the transpositions (i, j) for $i \neq j$ and $N(w) := \{\, (i, j) \mid i < j, w^{-1}(i) > w^{-1}(j) \,\}$ identifies with the set of inversions of w^{-1}. The root system Φ may be described as follows. Let S_n act in the natural way on $\mathbb{R}^n$ permuting coordinates, and let $\{\, e_1, \ldots, e_n \,\}$ be the standard ordered orthonormal basis. Then $\Phi := \{\, e_i - e_j \mid i \neq j \,\}$ with natural action by $W \times \{\pm 1\}$ and with $\Phi_+ := \{\, e_i - e_j \mid i < j \,\}$. One has $\Phi_w = \{\, e_i - e_j \mid i < j, w^{-1}(i) > w^{-1}(j) \,\}$ and $s_{e_i - e_j} = (i, j) \in T$.

The weak right order of W is the partial order $\leq$ on the set W defined by the condition $x \leq y$ if $\Phi_x \subseteq \Phi_y$ (equivalently, $N(x) \subseteq N(y)$ or $l(y) = l(x) + l(x^{-1}y)$). Weak right order is important in the basic combinatorics of W. For example, reduced expressions (that is expressions, $w = s_1 \cdots s_n$ with each $s_i \in S$ and $n = l(w)$) correspond bijectively to maximal chains in the order ideal of weak right order generated by w. Weak right order is a complete meet semilattice, which is a complete lattice if and only if W is finite (see [1]).

To explain the framework adopted in these papers, we discuss one of the main results in the context of Coxeter groups. Recall that a groupoid is a small category G in which every morphism is invertible. Let G be the groupoid with one object $\bullet$, with the Coxeter group W as automorphism group of $\bullet$. Let H be a non-empty groupoid, assumed connected (i.e. with at least one morphism between any two of its objects) for simplicity. The functor category G^H has as objects the functors $H \to G$, and as morphisms the natural transformations of such functors. It is itself a groupoid. It has a subgroupoid $K = G^H_\Box$ containing all objects of G^H, but only those morphisms $\nu\colon F \to F'$ (where $F, F'\colon H \to G$ are functors) such that for each morphism $h\colon a \to b$ in H,

$$\nu_b(\Phi_{F(h)}) = \Phi_{F'(h)}$$

where $\nu_b\colon F(b) \to F'(b)$ is the component of ν at b (note that $\nu_b, F(h), F'(h) \in \mathrm{Hom}_G(\bullet, \bullet) = W$). The groupoids studied in [4] are covering quotients of a very restricted class of components of groupoids $G^H_\Box$ (with their objects indexed by subsets of S instead of subsets of W) in cases in which H is connected and simply connected i.e. has a unique morphism between any two of its objects.

In general, the groupoids K have properties closely analogous to those of Coxeter groups mentioned above. For example, K has complete semilattice weak right orders, it has a canonical braid presentation specified by families of Coxeter matrices (which here give the lengths of the joins of pairs of simple generators) with additional combinatorial data, it has an abstract root system, and for a connected

groupoid L, one can form analogously $K^L_\square$ with similar properties, and so on recursively. Analogous facts hold for the groupoids of [4], where existence of canonical presentations was the main result and the other properties just mentioned were not established, for the Coxeter groupoids (with root systems) of [14], and for other natural classes of groupoids. The (components of) groupoids in [4] are not in general Coxeter groupoids, though they turn out to be so if W is finite.

These papers study a class of structures, which we call rootoids, which provides a natural framework within which the results mentioned in the previous paragraph, and many others, can be stated precisely and proved. The notion of rootoid is obtained by abstracting as an axiom the fact that the weak right order of a Coxeter group is a complete meet semilattice. (Note that the weak right order of a Coxeter group is not translation invariant, and that the theory of rootoids is quite distinct from the theory of ordered groups and groupoids). The notion of rootoid is far more general than that of Coxeter group, but special subclasses of rootoids have properties very similar to certain basic properties of Coxeter groups. The results stated above are proved by showing that rootoids are preserved by certain natural categorical constructions (for example, the category of rootoids has all small limits) which also preserve the conditions defining these subclasses.

Before discussing rootoids, the closely related notion of signed groupoid-sets, of which (W, Φ) is the archetypal example, will be described. The definition involves a category of signed sets, which has as objects sets S with a specified free action by $\{\pm 1\}$ and a specified set S_+ of $\{\pm 1\}$-orbit representatives, and as morphisms the $\{\pm 1\}$-equivariant functions. A signed groupoid-set is a pair $R = (G, \Psi)$ of a groupoid G and a representation Ψ of G in the category of signed sets. For $g\colon a \to b$ in G, define $\Psi_g := \Psi(b)_+ \cap \Psi(g)(-\Psi(a)_+)$. The set ${}_bG$ of morphisms of G with codomain b is preordered by $g \leq h$ if $\Psi_g \subseteq \Psi_h$. This gives one weak right preorder of R for each object b of G. Then R is said to be rootoidal if each weak right preorder is a complete meet semilattice satisfying a technical condition called JOP which implies that the weak right order embeds as an order ideal in a complete ortholattice. One says that R is principal if G has a set of generators (called simple generators) such that the minimal length of each morphism g as a product of these generators is equal to $|\Psi_g|$.

The above construction of the groupoid $K = G^H_\square$ can be adapted to the signed groupoid-set $R = (G, \Psi)$. An abstract root system of K is constructed as follows. First, there is a signed groupoid-set (K, Λ) where Λ is obtained by pullback of the functor Ψ under the evaluation homomorphism $\epsilon_b\colon K = G^H_\square \to G$ for some chosen object b of H. One forms a new signed groupoid-set $(K, \Lambda^{\mathrm{rec}})$ where the real compression Λ^{rec} is a sub-quotient representation of Λ. It is obtained, roughly speaking, by discarding any imaginary roots of Λ (defined as those $\alpha \in \Lambda(a)$ such that α and $\Lambda(g)(\alpha)$ have the same sign (positive or negative) for all morphisms g of G with domain a) and identifying any two remaining roots $\alpha, \beta \in \Lambda(a)$ which are equivalent in the equivalence relation associated to dominance preorder i.e. such that for all g with domain a, $\Lambda(g)(\alpha)$ and $\Lambda(g)(\beta)$ have the same sign.

The results about K referred to above in the case of Coxeter groups now follow from three facts. First, a Coxeter group gives rise to a principal, rootoidal, signed groupoid-set (W, Φ). Secondly, if $R = (G, \Psi)$ above is any principal, rootoidal, signed groupoid-set, then $(K, \Lambda^{\mathrm{rec}})$ is also a principal, rootoidal signed groupoid-set. Finally, the underlying groupoid of any principal, rootoidal signed groupoid-set has a braid presentation.

Rootoidal signed groupoid-sets provide one framework in which our main results may be formulated, and are particularly natural in certain cases (e.g related to Coxeter groups) where they can be realized concretely in real vector spaces. However, they seem less convenient in more abstract examples and for general categorical arguments. For these reasons, many results and arguments will be expressed in terms of analogues of the reflection cocycle N of W, especially since the concept of cocycle is already well established in general contexts and is very convenient for calculation. This approach leads to the notions of protorootoid and rootoid (explained later in the introduction) which will be used for our general development.

The previous discussion illustrates the three primary goals of these papers: to give examples of rootoids (of various classes), to describe general properties of rootoids (of various classes) and to study categories of rootoids (of various classes) and especially to establish their closure under certain natural categorical constructions. The third goal involves the most novel results, but the others are necessary to make these new results useful, and these aims will be pursued together in these papers. This first paper gives basic definitions, terminology and examples. The second paper will give more examples of rootoids and morphisms of rootoids, and discusses (without proof, and partly informally) some ideas, results and questions from subsequent papers of the series. These two papers together give the most rudimentary part of the theory, an introduction to the main ideas and results of subsequent papers and indications of ingredients of some of the main proofs. The rest of the theory, and the proofs, will appear in the later papers of the series.

The original motivations for this work were in relation to the set of initial sections of reflection orders of a Coxeter group W, which has significant applications in the combinatorics of Kazhdan-Lusztig polynomials and Bruhat order, and conjectural relevance to associated representation theory. Ordered by inclusion, the set of initial sections is a poset in which the weak order of W embeds naturally as an order ideal. Many of the main ideas of this work originated in the study in [9] of a longstanding conjecture which is equivalent to the statement that the set of initial sections is a complete ortholattice. Other papers which have suggested useful ideas include [19], [11], [12], [2], [6], [13], [14], and especially [4]. Though the theory described in this paper and its sequels has been largely motivated by study of Coxeter groups, the most novel aspects of the work involve the interaction of several partly independent formalisms of a quite general nature and it seems unlikely that rootoids and protorootoids provide the natural setting for all these formalisms. Limitations of space and energy preclude detailed discussion of all the known generalizations and variations, but many are indicated in the remarks.

Finally, we mention that formulations and proofs of many of the main results in these papers require only knowledge of rather limited basic aspects of algebra, combinatorics of ordered sets and category theory as can be found in, for example, [17], [7] and [18]. More background knowledge, and especially familiarity with Coxeter groups, is assumed for the examples and applications. As general references on Coxeter groups, see [3], [16] and [1].

Introduction to this paper. The concerns of this first paper are as follows. Section 1 gives background and fixes frequently used general notation. Section 2 defines protorootoids, which provide a convenient general context for this work. A protorootoid is a triple $\mathscr{R} := (G, \Lambda, N)$ of a groupoid G, a representation Λ of G in the category of Boolean rings (i.e. a functor $\Lambda\colon \Gamma \to \mathbf{BRng}$) and a cocycle $N \in Z^1(G, \Lambda)$ for the representation of G underlying Λ in $\mathbb{Z}$-mod. Concretely, N is a function which associates to each morphism g of G, with codomain a, say, an element $N(g)$ of the Boolean ring $\Lambda(a)$, satisfying the cocycle condition $N(gg') = N(g) + gN(g')$ where we abbreviate $gN(g') := (\Lambda(g))(N(g'))$. The motivating examples are the protorootoids $\mathscr{C}_{(W,S)} := (W, \Lambda, N)$ attached to Coxeter systems (W, S). In this, Λ denotes the conjugacy representation of W on the Boolean ring $\wp(T)$ of subsets of the reflections T, and $N\colon W \to \wp(T)$ is the reflection cocycle of (W, S).

Associated to the protorootoid $\mathscr{R}$ is a family of weak right preorders indexed by the objects of G; the set ${}_aG$ of morphisms of G with fixed codomain a is preordered by $x \le y$ if $N(x) \le N(y)$ in the natural partial order of the Boolean ring $\Lambda(a)$ (which is defined by $r \le r'$ if $r = rr'$). There is a natural category of protorootoids in which, in particular, morphisms induce preorder preserving maps of associated weak right preorders. Every protorootoid (G, Λ, N) has an underlying groupoid-preorder, forgetting Λ and N but remembering the groupoid G and its weak right preorders; subsequent papers will show that an alternative approach to much of the theory of protorootoids and rootoids can be developed in terms of groupoid-preorders.

Important subclasses of protorootoids are defined in Section 3. These include the faithful ones (for which weak right preorders are partial orders), complete faithful ones (the weak right orders of which are complete lattices), the faithful, interval finite ones (for which intervals in weak right order are finite, and for which the set of atoms of weak right orders is a groupoid generating set) and the principal ones (for which the underlying groupoid has a set of simple generators with respect to which the length of any morphism $g\colon b \to a$ is equal to the rank in $\Lambda(a)$ of $N(g)$ i.e the length of a maximal chain from 0 to $N(g)$). The class of preprincipal protorootoids is defined by requiring, in an interval finite, faithful protorootoid, a natural generalization of a length-compatibility condition from [19] (expressed as a cocycle compatibility condition in [11], [12]) which requires that $l_S(w's') = l_S(w') \pm l_S(s')$ for all $w \in W'$, $s' \in S'$ for special embeddings $W' \subseteq W$ of Coxeter groups, where S and S' are the sets of simple generators of W and W' respectively.

The abridgement of a protorootoid $\mathscr{R} = (G, \Lambda, N)$ is defined by $\mathscr{R}^a := (G, \Lambda', N')$ where Λ' is the G-subrepresentation of Λ (in the category of Boolean rings) generated by the values of the cocycle N, and N' is the evident restriction of N. It gives a minimal version of $\mathscr{R}$ preserving G and its weak right preorders. Abridgement is an

analogue for protorootoids of real compression, but has better functorial properties. A technically important fact proved here is that a protorootoid is preprincipal if and only if its abridgement is principal.

In Section 4, a rootoid is defined to be a faithful protorootoid in which the weak orders are complete meet semilattices which satisfy a condition called JOP (join orthogonality property). To explain JOP, say that two morphisms x, y with common codomain a are orthogonal if $N(x)N(y) = 0$ in $\Lambda(a)$ (or $x^{-1} \leq x^{-1}y$, in terms of weak right preorder). Then JOP requires that if each of a non-empty family of morphisms (x_i) is orthogonal to y and (x_i) has a join (least upper bound) x, then x is orthogonal to y.

Section 4 also contains the definition of the category of rootoids. Consider two rootoids and a morphism of protorootoids between them such that each induced map θ of weak orders preserves all meets (greatest lower bounds) and joins of non-empty sets which exist in its domain. Each θ then has a partially defined (categorical) left adjoint $\theta^{\perp}$, defined on the order ideal generated by the image of θ. The morphism is defined to be a morphism of rootoids if it satisfies the following adjunction orthogonality property (AOP): for all θ, one has that $\theta(x)$ and y (in the order ideal) are orthogonal if and only if x and $\theta^{\perp}(y)$ are orthogonal (in this, "if" holds automatically). A subcategory of the category of rootoids which is important in relation to functor rootoids is that of rootoid local embeddings, in which morphisms are restricted so that the weak orders of the domain embed as join-closed meet subsemilattices of the weak orders of the codomain.

The principal rootoids have many basic properties in common with Coxeter groups and their study is one main focus of these papers (study of complete rootoids is another). However, in categorical contexts it is technically more convenient to work with preprincipal rootoids. The JOP ensures that several categorical constructions with protorootoids preserve rootoids and that the morphisms involved induce semilattice homomorphisms between corresponding weak right orders. The AOP ensures further that many such constructions preserve preprincipal rootoids; composing such constructions with abridgement gives constructions preserving principal rootoids. Though we do not emphasize it in the introductory papers, many results of a categorical nature about principal rootoids hold for a larger class (of regular, saturated, pseudoprincipal rootoids) in which the principal rootoids are distinguished as the interval finite members. See Section 3 for the definitions, which are motivated partly by conjectures on initial sections of reflection orders of Coxeter groups.

Any signed groupoid-set (G, Ψ) gives rise to a protorootoid $(G, \wp_G(\Psi/\{\pm 1\}), M)$ where the 1-cocycle $M \in Z^1(G, \wp_G(\Psi/\{\pm 1\}))$ classifies Ψ as a principal bundle with fiber $\{\pm 1\}$ over a suitably defined quotient representation $\Psi/\{\pm 1\}$ of Ψ, and $\wp_G$ is a power-set functor from G-sets to G-Boolean rings. The triple $(G, \Psi/\{\pm 1\}, M)$ is called a set protorootoid. In Section 5, categories of set protorootoids and signed groupoid-sets are defined. It is shown that the construction above induces an equivalence between these two categories, and the category of set protorootoids is trivially equivalent to a subcategory of the category of protorootoids. These facts will be supplemented by a result in subsequent papers showing that protorootoids have

representations as set protorootoids (in a similar manner as Boolean rings have representations as rings of sets, by Stone's theorem). Together (see 5.8), these results make it possible to restate many of the results of these papers in terms of signed groupoid-sets and then to obtain stronger analogues of some of them for special classes of signed groupoid-sets which have realizations in real vector spaces (there are several natural notions of realization, of different strength and generality).

Section 6 recalls some background on Coxeter groups, and discusses basic examples of rootoids. The reader may wish to read it in parallel with earlier sections. The protorootoid $\mathscr{C}_{(W,S)}$ of a Coxeter system is shown to be a principal rootoid, complete if and only if W is finite. Two proofs are given; the more self-contained one involves a useful semilocal criterion (SLC) for an interval finite, faithful protorootoid to be a rootoid. Protorootoids are also defined in Section 6 from finite, real, central hyperplane arrangements and shown using [2] to be rootoids if and only if the arrangement is simplicial, in which case the rootoid is complete and principal. Similarly, Coxeter groupoids and Weyl groupoids with root systems in the sense of [6] and [14] give rise to principal rootoids (complete if and only if finite, in the case of a connected Coxeter groupoid) and protorootoids naturally attached to simplicial oriented geometries (see [2]) are rootoids if and only if the oriented geometry is simplicial (in which case the rootoid is complete and principal). Discussion of these and most other important examples is deferred to subsequent papers, though for diversity, rootoids with the additive group of real numbers or a compact real orthogonal group as underlying groupoid are also described in Section 6.

Finally in this paper, a brief Section 7 shows that each weak order of a complete rootoid has a longest element (analogous to that of a finite Coxeter group) making it a complete ortholattice. These longest elements are shown to give rise to an involutory automorphism of the rootoid (the analogue of the diagram automorphism given by conjugation by the longest element of a finite Coxeter group).

1. Background, notation and conventions

1.1. **Ordered sets.** The following terminology and notation for posets, lattices and semilattices will be used throughout this series of papers.

A *poset* (i.e. partially ordered set) $(\mathscr{L}, \leq)$ will usually be denoted just as $\mathscr{L}$ (note that the empty poset $\emptyset$ is permitted). An *order ideal* of a poset $(\Lambda, \leq)$ is a subset $\Gamma \subseteq \Lambda$ such that if $x, y \in \Lambda$ satisfy $x \leq y \in \Gamma$, then $x \in \Gamma$. *Order coideals* Γ are defined dually: if $x \geq y \in \Gamma$, then $x \in \Gamma$. Closed intervals in Λ are denoted as $[x, y] = [x, y]_\Lambda := \{\, z \in \Lambda \mid x \leq z \leq y \,\}$. An element x is an *atom* of Λ if Λ has a minimum element, say m, and $|[m, x]| = 2$ (generally, the cardinality of a set A is denoted by $|A|$). *Coatoms* are defined dually.

The category of posets and order preserving maps is denoted **Ord**. It is a full, reflective subcategory of the category **PreOrd** of preordered sets and preorder preserving maps (by definition, a *reflective* (resp., *coreflective*) subcategory of a category is one such that the corresponding inclusion functor has a left (resp., right) adjoint). Recall that a *preordered set* $(L, \leq)$ is a set L equipped with a reflexive and transitive relation $\leq$. The left adjoint **PreOrd** $\to$ **Ord** sends the preordered set $(L, \leq)$ to its

associated poset, which is defined as follows. The associated equivalence relation $\sim$ of $(L, \leq)$ is the relation on L given by $a \sim b$ if $a \leq b$ and $b \leq a$. Denote the equivalence class of $a \in L$ as $[a]_L$ or $[a]$ for short. Then the associated poset $(L/\!\sim, \preceq)$ of $(L, \leq)$ has $L/\!\sim := \{\, [a] \mid a \in L \,\}$ and $[a] \preceq [b]$ if and only if $a \leq b$.

A preordered set (or partial order) is *directed* if all of its finite subsets have an upper bound (in particular, it is non-empty). A subset of a preorder is directed if it is directed in the induced preorder. A subset X of a preordered set (or poset) $(L, \leq)$ is *cofinal* if for any $y \in L$, there exists $x \in X$ with $y \leq x$.

In any partially ordered set, the *meet* or greatest lower bound of a subset Γ (resp., the *join* or least upper bound of the subset Γ) is denoted by $\bigwedge \Gamma$ (resp., $\bigvee \Gamma$) if it exists. Notions of meets and joins are defined similarly for preordered sets (but are only determined up to equivalence when they exist). Write $\bigwedge\{\gamma_1, \ldots, \gamma_n\} = \gamma_1 \wedge \cdots \wedge \gamma_n$ and $\bigvee\{\gamma_1, \ldots, \gamma_n\} = \gamma_1 \vee \cdots \vee \gamma_n$ for joins and meets of finite sets. Notation such as $\bigwedge_i x_i$, $\wedge x_i$ and $\bigvee_i x_i$, $\vee x_i$ will be used for meets and joins of indexed families $\{x_i\}_{i \in I}$. A *complete lattice* $\mathscr{L}$ is a non-empty partially ordered set in which every subset has a meet and join; in particular, $\mathscr{L}$ has a minimum element $0_{\mathscr{L}}$ and a maximum element $1_{\mathscr{L}}$. A *complete ortholattice* is a complete lattice $\mathscr{L}$ equipped with a given order reversing map $x \mapsto x^{\complement} \colon \mathscr{L} \to \mathscr{L}$, called *orthocomplementation*, such that for all $x \in \mathscr{L}$, $(x^{\complement})^{\complement} = x$, $x \wedge x^{\complement} = 0_{\mathscr{L}}$ and $x \vee x^{\complement} = 1_{\mathscr{L}}$.

A *complete meet semilattice* is a possibly empty, partially ordered set $\mathscr{L}$ in which any non-empty subset Z has a meet $\bigwedge Z$. This implies that any subset Z of $\mathscr{L}$ which is bounded above in $\mathscr{L}$ has a join $\bigvee Z$ in $\mathscr{L}$ (namely, the meet in $\mathscr{L}$ of the set of upper bounds of Z in $\mathscr{L}$), and that $\mathscr{L}$ has a minimum element if it is non-empty. A *complete join-closed meet subsemilattice* Z of a complete meet semilattice $\mathscr{L}$ is defined to be a subset Z of $\mathscr{L}$ which is closed under formation of arbitrary meets in $\mathscr{L}$ of non-empty subsets of Z, and such that for any non-empty subset of Z with a join in $\mathscr{L}$, that join is an element of Z. Such a subset Z is itself a complete meet semilattice, with meets (and those joins which exist) of its non-empty subsets coinciding with the meets (and joins) of those subsets in $\mathscr{L}$. Note that it is not required that the minimum element of a non-empty complete join-closed meet subsemilattice of $\mathscr{L}$ must coincide with the minimum element of $\mathscr{L}$. Dually, *complete join semilattices* and their *complete meet-closed join subsemilattices* are defined.

1.2. **Boolean rings.** Some of the following terminology is slightly non-standard (cf. [7]), but it is convenient for our purposes here. An element of a monoid or ring is *idempotent* if $x^2 = x$. A *Boolean ring* is a (possibly non-unital) ring in which every element is idempotent. A Boolean ring B is commutative and satisfies $x + x = 0$ for all $x \in B$. Also, B has a partial order $\leq$ defined by $x \leq y$ if $xy = x$, for $x, y \in B$. It will usually be more convenient to denote this partial order as $\subseteq$, the multiplication as $(x, y) \mapsto x \cap y$ and the addition as $(x, y) \mapsto x + y$. The join $x \vee y$ of two elements x, y of B always exists; it is given by $x \vee y = x + y + xy$ and will be denoted as $x \cup y$. The meet of x and y exists and is their product $x \cap y$. The additive identity 0 of B may be denoted as $\emptyset$. A unital Boolean ring B is called a *Boolean algebra*; in any such Boolean algebra, the *complement* $x^{\complement}$ of $x \in B$ is defined by $x^{\complement} := 1_B + x$. This

notation is in accord with the standard notation for the Boolean algebra $\wp(A)$ arising as the power set of a set A, where $\subseteq$ is inclusion, $\cap$ is intersection, $+$ is symmetric difference $x+y=(x\cup y)\setminus(x\cap y)$, $x^{\complement}=A\setminus x$, and $\cup$ is union. It is partially justified in general by the existence (by Stone's theorem) of an isomorphism of any Boolean ring with a subring of some Boolean algebra $\wp(A)$. The field $\mathbb{F}_2$ of two elements is an important example of a Boolean algebra.

Remarks. The use of $\cup, \cap, \subseteq$ for Boolean rings enables us to reserve $\vee, \wedge, \leq$ mostly for weak (right) orders.

1.3. The category **BRng** of Boolean rings has Boolean rings as objects, and its morphisms are ring homomorphisms, with usual composition. The category **BAlg** of Boolean algebras is the subcategory of **BRng** with Boolean algebras as objects, and ring homomorphisms which preserve identity elements as morphisms.

There is a contravariant power set functor $\wp\colon \mathbf{Set}\to\mathbf{BRng}$ such that for any set A, $\wp(A)$ is the power set of A, regarded as Boolean ring as above and for $f\colon A\to A'$, $\wp(f)\colon \wp(A')\to\wp(A)$ is the map $x\mapsto f^{-1}(x):=\{\, a\in A \mid f(a)\in x\,\}$ for $x\subseteq A'$, which is a homomorphism of Boolean rings. The functor $\wp$ factors through the inclusion $\mathbf{BAlg}\to\mathbf{BRng}$.

1.4. **Categories, morphisms, stars.** For any small category G, the sets of objects and morphisms of G are denoted as $\mathrm{ob}(G)$ and $\mathrm{mor}(G)$ respectively. The domain (resp., codomain) of a morphism α in G is denoted $\mathrm{dom}(\alpha)$ (resp., $\mathrm{cod}(\alpha)$). For any $a,b\in\mathrm{ob}(G)$, define ${}_aG_b:=\mathrm{Hom}_G(b,a)$, the *left star* ${}_aG:=\dot\cup_{b\in\mathrm{ob}(G)}\,{}_aG_b$ of G at a and the *right star* $G_b:=\dot\cup_{a\in\mathrm{ob}(G)}\,{}_aG_b$ of G at b. In these equations and elsewhere, the symbol "$\dot\cup$" emphasizes that a union is one of pairwise disjoint sets. In a Boolean ring, $z=\dot\cup_{i=1}^{n}z_i$ would mean similarly that z is the join of $z_1,\dots,z_n$ where $z_i\cap z_j=\emptyset$ (i.e. $z_iz_j=0$) for $i\neq j$. To simplify terminology, left stars will be simply called stars when they are referred to other than notationally.

For any subset X of $\mathrm{mor}(G)$ and for all $a,b\in\mathrm{ob}(G)$, set ${}_aX:=X\cap{}_aG$, $X_b:=X\cap G_b$ and ${}_aX_b:=X\cap{}_aG_b$. If a product $g_1\dots g_n$ of morphisms g_i of G appears in a formula (such as $g=g_1\dots g_n$) it is tacitly required that the g_i are such that the composite is defined. Occasionally, the notation $\exists g_1\cdots g_n$ is used to indicate that a composite $g_1\dots g_n$ is defined.

For two categories G, C with G small, let C^G denote the category of functors $G\to C$, with natural transformations between such functors as morphisms.

1.5. **Slice category.** Given a category G and object $a\in\mathrm{ob}(G)$, there is a *slice category* G/a with objects the morphisms $f\colon b\to a$ in G. A morphism $F\colon f\to f'$ in G/a, where $f'\colon b'\to a$ in G, is a morphism $F\colon b\to b'$ in G such that $f=f'F$, and composition is induced by composition in G. There is a functor $G/a\to G$ given on objects by $f\mapsto b$ and on morphisms by mapping $F\colon f\to f'$ in G/a to $F\colon b\to b'$ in G, with notation as above. Dually, define the *coslice category* $a\backslash G$.

1.6. **Opposites.** The *opposite* of a category (resp., preorder, poset, ring etc) G is denoted as G^{op}.

1.7. **Representations.** A *representation* of a (small) category G in a category X is a functor $F\colon G \to X$. Two representations are equivalent if they are naturally isomorphic as functors. A concrete category is a category X equipped with a faithful functor $U\colon X \to \mathbf{Set}$ where $\mathbf{Set}$ is the category of sets and functions; examples include $\mathbf{BRng}$ and $\mathbf{BAlg}$ (with U the underlying set functor) and $\mathbf{Set}$ (with U the identity functor). For certain well-known concrete categories, such as those above, we will often not distinguish notationally between an object or morphism of X, and its image under U, as is customary.

For a representation F of a category G in a concrete category X, $U(F(x))$ may be abbreviated as ${}_xF$ for $x \in \mathrm{ob}(G)$, and $(U(F(g)))(a) \in {}_yF$ may be abbreviated as ga or $g(a)$ for $g\colon x \to y$ in $\mathrm{mor}(G)$ and $a \in {}_xF$. It will often be tacitly assumed, replacing F by an equivalent functor if necessary that the sets ${}_xF$ for $x \in \mathrm{ob}(G)$ are pairwise disjoint. By a *subrepresentation* of F, we shall mean a representation F' of G in X such that there is a natural transformation $\mu\colon F' \to F$ such that all functions $U(\mu_x)\colon UF'(x) \to UF(x)$ for $x \in \mathrm{ob}(G)$ are inclusion maps. Define a *trivial representation* to be a representation that is equivalent to a constant functor.

1.8. **Groupoids.** A groupoid G is a small category in which every morphism is invertible. The contravariant self-equivalence $\iota_G\colon G \to G$ of the groupoid G induced by inversion will be denoted as $x \mapsto x^*$ for short. Thus, $x^* := x$ if $x \in \mathrm{ob}(G)$ and $x^* := x^{-1}$ if $x \in \mathrm{mor}(G)$. The automorphism groups ${}_aG_a = \mathrm{Aut}_G(a)$ of objects a of G are called the *vertex groups* of G. The identity morphism of G at a will be denoted 1_a or ${}_a1$. The morphisms g of G which are in some (necessarily unique) vertex group (i.e. such that $\exists gg$) are called *self-composable morphisms*. A groupoid is empty if its sets of objects and morphisms are empty, and trivial if it has one object and one morphism. Groups will be regarded in the usual way or as groupoids with one object, as convenience dictates.

Given a representation $\Lambda\colon G \to \mathbf{Set}$ of a groupoid G in the category of sets, there is a corresponding representation $\wp_G(\Lambda)\colon G \to \mathbf{BRng}$ of G in the category of Boolean rings, defined by $\wp_G(\Lambda) := \wp\Lambda\iota_G,$. This has a subrepresentation $\wp'_G(\Lambda)$ such that for each $a \in \mathrm{ob}(G)$, ${}_a(\wp'_G(\Lambda))$ is the subring of ${}_a(\wp_G(\Lambda))$ consisting of all finite subsets. Note that $\wp_G$ may be regarded as a contravariant functor $\wp_G\colon \mathbf{Set}^G \to \mathbf{BRng}^G$: if $\nu\colon \Lambda \to \Gamma$ is a natural transformation between functors $\Lambda, \Gamma\colon G \to \mathbf{Set}$, then $\wp_G(\nu)\colon \wp_G(\Gamma) \to \wp_G(\Lambda)$ is the natural transformation with component at $a \in \mathrm{ob}(G)$ given by $(\wp_G(\nu))_a = \wp(\nu_a)\colon \wp(\Gamma(a)) \to \wp(\Lambda(a))$.

1.9. **Connectedness.** A non-empty groupoid G is said to be *connected* (resp., *simply connected*) if there is at least one (resp., at most one) morphism between any two of its objects. By convention, the empty groupoid is simply connected but not connected (note that some sources regard the empty groupoid as connected). A connected, simply connected groupoid G is determined up to isomorphism by the cardinality $|\,\mathrm{ob}(G)|$ of the set of its objects, which can be any non-zero cardinal.

1.10. **The category of groupoids.** Let $\mathbf{Cat}$ denote the category of small categories with functors as morphisms, with usual composition of functors. For categories G, H, $\mathrm{Hom}_{\mathbf{Cat}}(G, H)$ is the set of objects of a category, in which a morphism

$\nu\colon F \to F'$ is a natural transformation $\nu\colon F \to F'$ of functors $G \to H$. For a morphism $\nu'\colon F' \to F''$, the composite $\nu'\nu$ has components $(\nu'\nu)_a = \nu'_a\nu_a$ for all $a \in \mathrm{ob}(G)$. For our purposes, there is little need for, and we do not adopt, 2-categorical language, but basic facts (such as the interchange law) about composition of functors and natural transformation will be used without comment.

The category **Gpd** of groupoids is the full subcategory of **Cat** which has groupoids as objects. For groupoids G, H, the category $\mathrm{Hom}_{\mathbf{Gpd}}(G, H)$ is a groupoid; the inverse $\nu^*\colon F' \to F$ of a natural transformation $\nu\colon F \to F'$, where $F, F'\colon G \to H$, has components $(\nu^*)_a = \nu^*_a := (\nu_a)^*$. Both **Cat** and **Gpd** are complete and cocomplete i.e. have all limits and colimits of functors from small categories. See [15] and [5] as general references on groupoids, though our terminology and notation differ from theirs.

1.11. **Components.** A *subgroupoid* of a groupoid G is a subcategory H of G which contains the inverse in G of any morphism of H. A non-empty subgroupoid H of G is called a *component* of G if it is maximal connected subgroupoid of G (i.e. if $\mathrm{mor}(H)$ is maximal under inclusion amongst morphism sets of connected subgroupoids of G). A component is full as a subcategory. The set of morphisms (resp., objects) of G is the disjoint union of the sets of morphisms (resp., objects) of its components For an object or morphism x of G, $G[x]$ denotes the component of G containing x. The study of any groupoid largely reduces to that of its components.

1.12. **Coverings.** A morphism $\pi\colon H \to G$ in **Gpd** is called a *covering morphism* or *covering* of G if the induced maps ${}_a\pi : {}_aH \xrightarrow{\cong} {}_{\pi(a)}G$ given by $h \mapsto \alpha(h)$ are bijections for all $a \in \mathrm{ob}(H)$. For such a covering, each component of H is mapped into a component of G; the restriction to a morphism between two components is surjective on both objects and morphisms. For example, the natural embedding of a connected component H of G as a subgroupoid of G is a covering morphism. For a fixed groupoid G, the category of *covering transformations* or *coverings* of G is the subcategory of the slice category $\mathbf{Gpd}/G$ with objects the coverings of G, and only those morphisms which are isomorphisms. Morphisms in the category of coverings of G are called *covering transformations.* If a covering π as above is surjective on objects, we call G a *covering quotient* of H and π a *covering quotient morphism.*

By definition, a *universal covering groupoid* of a groupoid G is a groupoid H equipped with a covering morphism $\pi\colon H \to G$, called the *universal covering morphism*, such that that H is simply connected and the natural map from components of H to components of G induced by π is bijective. The universal covering morphism exists and is determined up to isomorphism as an object of the category of covering morphisms of G. It is a covering quotient morphism.

The construction of the universal covering $\pi\colon H \to G$ in general readily reduces to the case G is connected, in which case, π may be identified with the natural morphism $\pi\colon G/a \to G$ for any $a \in \mathrm{ob}(G)$. See [15] for basic facts about coverings.

1.13. **Groupoid 1-cocycles.** Let $\Psi\colon G \to \mathbb{Z}\text{-mod}$ be a fixed representation of a groupoid G in the category of abelian groups. A 1-*cocycle* N of G for Ψ is a family

of maps ${}_aN\colon {}_aG \to {}_a\Psi$ for $a \in \mathrm{ob}(G)$ satisfying the *cocycle condition*

$$ {}_aN(gh) = {}_aN(g) + \Lambda(g)({}_bN(h)) \tag{1.13.1}$$

for all $a, b \in \mathrm{ob}(G)$, $g \in {}_aG_b$ and $h \in {}_bG$. Unless confusion is likely, N is regarded as a function $\mathrm{mor}(G) \to \cup_{a\in\mathrm{ob}(G)}\, {}_a\Psi$, and ${}_aN(g)$ is denoted simply as $N(g)$ or sometimes even N_g. Abbreviate $(\Psi(g))(x)$ as gx for $g \in G_a$ and $x \in \Psi(a)$. Then (1.13.1) states that for $g, h \in \mathrm{mor}(G)$,

$$N(gh) = N(g) + gN(h). \tag{1.13.2}$$

For example, given a family $(x_a)_{a\in\mathrm{ob}(G)}$ with $x_a \in {}_a\Psi$, there is a cocycle N defined by $N(g) = x_a - g(x_b)$ for $g \in {}_aG_b$ and all $a, b \in \mathrm{ob}(G)$; a cocycle of this type is called a *coboundary*. The cocycles and coboundaries under their natural pointwise addition form additive abelian groups $Z^1(G, \Psi) \supseteq B^1(G, \Psi)$ and the quotient abelian group

$$H^1(G, \Psi) := Z^1(G, \Psi)/B^1(G, \Psi) \tag{1.13.3}$$

is called the *first cohomology group*.

The cocycle condition readily implies that for any cocycle N of G and any identity morphism 1_a of G, one has $N(1_a) = 0$ and for any morphism g of G, $N(g^*) = g^*N(g)$.

Given a cocycle $N \in Z^1(G, \Psi)$ and a groupoid homomorphism $\alpha\colon H \to G$, there is a pullback representation $\Psi\alpha\colon H \to \mathbb{Z}\text{-mod}$ of H and a cocycle $N' \in Z^1(H, \Psi\alpha)$ defined by $N'(h) = N(\alpha(h))$. By abuse of notation, N' will often be denoted as $N' = N\alpha$. Similarly, given a natural transformation $\nu\colon \Psi \to \Psi'$ between functors $\Psi, \Psi'\colon G \to \mathbb{Z}\text{-mod}$ and $N \in Z^1(G, \Psi)$, denote by $\nu N := N''$ the cocycle $N'' \in Z^1(G, \Psi')$ defined by ${}_aN''(g) = \nu_a({}_aN(g))$ for all $a \in \mathrm{ob}(G)$.

Associated to the G-cocycle N on Ψ, there is a representation $\Lambda\colon G \to \mathbb{Z}\text{-mod}$ given by setting $\Lambda(x)$ equal to $\Psi(x)$ for any object x of G, and $(\Lambda(g))(x) = N(g)+gx$ for $g \in G_a$ and $x \in {}_a\Psi$. This will be called the N-twisted G-action or *dot action*, and denoted

$$(g, x) \mapsto (\Lambda(g))(x) = g \cdot x := N(g) + gx. \tag{1.13.4}$$

Remarks. A representation $\Psi\colon G \to \mathbf{BRng}$ gives, by applying the forgetful functor $\mathbf{BRng} \to \mathbb{Z}\text{-mod}$, an underlying representation $\Psi_{\mathbb{Z}}\colon G \to \mathbb{Z}\text{-mod}$. Similarly for morphisms (natural transformations) between such representations. By a cocycle N for a representation $\Psi\colon G \to \mathbf{BRng}$, we shall mean a 1-cocycle N for the underlying representation $\Psi_{\mathbb{Z}}\colon G \to \mathbb{Z}\text{-mod}$. Analogous conventions are used for representations $\Psi\colon G \to \mathbf{BAlg}$.

It is particularly important for our applications here that the values $N(g)$, $g \in {}_aG$ of the cocycle N are naturally partially ordered (since they lie in the Boolean ring $\Psi(a)$). There are other notions of 1-cocycle which can be used in formulating similar theories (Remark 5.5(2)–(3) suggests one of them).

1.14. The following facts are well-known, and their simple proofs are omitted.

Lemma. *Let G be a connected, simply connected groupoid.*

(a) *Any representation of G is equivalent to a trivial representation (i.e. a constant functor).*

(b) *If* $\Psi\colon G \to \mathbb{Z}$-mod *is a representation of* G, *then* $H^1(G,\Psi) = 0$ *i.e. any* G-*cocycle for* Ψ *is a coboundary.*

2. Protorootoids

2.1. **Protorootoids.** Protorootoids are important in these papers mainly as a framework in which to define and study rootoids. However, it is technically convenient to indicate in the exposition many properties of rootoids which hold even for protorootoids, and to give definitions applicable to both protorootoids and rootoids when possible.

Definition. A protorootoid is a triple (G,Λ,N) such that G is a groupoid, $\Lambda\colon G \to \mathbf{BRng}$ is a representation of G in the category $\mathbf{BRng}$ and $N \in Z^1(G,\Lambda)$ is a G-cocycle for Λ.

A protorootoid (G,Λ,N) is said to be unitary if Λ factors as a composite functor $G \to \mathbf{BAlg} \to \mathbf{BRng}$ where $\mathbf{BAlg} \to \mathbf{BRng}$ is the inclusion functor.

Remarks. (1) Note that given a groupoid G and functor $\Lambda\colon G \to \mathbf{BRng}$, (G,Λ,N) is a protorootoid for any $N \in Z^1(G,\Lambda))$. In particular, even for fixed (G,Λ), protorootoids (G,Λ,N) may exist in abundance (since $Z^1(G,\Lambda) \supseteq B^1(G,\Lambda))$.

(2) Despite our use of $\emptyset$ or 0 to denote the additive identity of ${}_a\Lambda$ for each $a \in \mathrm{ob}(G)$, it is tacitly assumed that the sets ${}_a\Lambda$ are pairwise disjoint (cf. 1.7).

2.2. **Weak order.** Let $\mathscr{R} = (G,\Lambda,N)$ be a protorootoid. For $a \in \mathrm{ob}(G)$, define the subposet ${}_a\mathscr{L} := \{\, N(g) \mid g \in {}_aG \,\}$ of ${}_a\Lambda$. The poset ${}_a\mathscr{L}$ is called the *weak order* (of $\mathscr{R}$, or G) at a. The weak order at a has a minimum element, denoted as 0 or $\emptyset$. Those joins and meets which exist in ${}_a\mathscr{L}$ will be denoted using $\vee$ or $\bigvee$, and $\wedge$ or $\bigwedge$, respectively.

Define the coproduct $\mathscr{L} := \coprod_{a\in\mathrm{ob}(G)} {}_a\mathscr{L}$ in $\mathbf{Ord}$ and denote its partial order still as $\subseteq$. In view of our assumption that the sets ${}_a\Lambda$ are pairwise disjoint, $\mathscr{L}$ is identified with the disjoint union $\mathscr{L} := \dot\bigcup_{a\in\mathrm{ob}(G)} {}_a\mathscr{L}$ as set, with the partial order which restricts to the weak order on ${}_a\mathscr{L}$ and such that no element of ${}_a\mathscr{L}$ is comparable with any element of ${}_b\mathscr{L}$ for any $a \neq b$. The poset $(\mathscr{L},\subseteq)$ will be called the *big weak order* of $\mathscr{R}$. Note that it is stable under the dot action of G in the following sense: if $A = N(g) \in {}_a\mathscr{L}$ where $g \in {}_aG$ and $h \in {}_bG_a$, then

$$h \cdot A = h \cdot N(g) = N(h) + hN(g) = N(hg) \in {}_b\mathscr{L}. \tag{2.2.1}$$

However, the dot G-action is not order-preserving (in a similar sense) on $\mathscr{L}$ in general. On the other hand, the action of G via Λ is by (necessarily order-preserving) isomorphisms between the Boolean rings ${}_a\Lambda$, which do not in general preserve their subsets ${}_a\mathscr{L}$.

Comparability of two elements $N(x), N(z)$ in ${}_a\mathscr{L}$ admits several equivalent reformulations. In fact, let $x,y,z \in \mathrm{mor}(G)$ with $z = xy$. Then $N(z) = N(x) + xN(y)$ and $N(z^*) = N(y^*) + y^*N(x^*)$. Hence

$$\begin{aligned} N(x) \subseteq N(z) \iff N(x) \cap xN(y) = \emptyset \iff N(x^*) \cap N(y) = \emptyset \\ \iff N(y^*) \cap y^*N(x^*) = \emptyset \iff N(y^*) \subseteq N(z^*). \end{aligned} \tag{2.2.2}$$

2.3. **Weak right preorder.** There is a preorder ${}_a{\le}$ on ${}_aG$, called the *weak right preorder* at a (or on ${}_aG$), given by $x \,{}_a{\le}\, y$ if $N(x) \subseteq N(y)$. By definition, there is a preorder preserving map

$$x \mapsto N(x) \colon {}_aG \to {}_a\mathscr{L}. \tag{2.3.1}$$

This map identifies with the component at $({}_aG, {}_a{\le})$ of the unit of the adjunction arising from existence of a left adjoint of **Ord** $\to$ **PreOrd** (see 1.1).

Define the *big right weak preorder* as the coproduct $\coprod_{a\in \mathrm{ob}\,(G)}({}_aG, {}_a{\le})$ in **PreOrd**. Its underlying set is taken to be $\dot\bigcup_{a\in\mathrm{ob}(G)}{}_aG = \mathrm{mor}(G)$, and the partial order on it is denoted $\le$. Two morphisms with distinct codomains are incomparable in $\le$, and the restriction of $\le$ to ${}_aG$ is ${}_a{\le}$. Similarly, define the *weak left preorder* $\le_a$ at a as the preorder of G_a defined by $x \le_a y$ if and only if $x^* \,{}_a{\le}\, y^*$, etc.

2.4. **Properties of weak right preorders.** Some useful properties of weak right preorders of protorootoids are listed in the next proposition.

Proposition. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid. Let $a, b, c, d \in \mathrm{ob}(G)$, $v \in {}_aG_d$, $x \in {}_aG_b$, $y \in {}_bG_c$ and $w \in {}_bG$. Then*

(a) $1_a \,{}_a{\le}\, x$.
(b) *If* $x \,{}_a{\le}\, xy$ *then* $y^* \,{}_c{\le}\, y^*x^*$.
(c) *If* $x \,{}_a{\le}\, xy$ *and* $x \,{}_a{\le}\, xw$, *then* $xy \,{}_a{\le}\, xw$ *if and only if* $y \,{}_b{\le}\, w$.
(d) *If* $v^* \,{}_d{\le}\, v^*x$, $v \,{}_a{\le}\, xy$ *and* $y^* \,{}_c{\le}\, y^*w$, *then* $v^* \,{}_d{\le}\, v^*xw$.
(e) *If* $y \,{}_b{\le}\, w$ *and* $w \,{}_b{\le}\, y$ *then* $xy \,{}_a{\le}\, xw$.

Proof. Part (a) is trivial and (b) follows from (2.2.2). Part (c) amounts to the fact that if $N(x) \cap xN(y) = \emptyset = N(x) \cap xN(w)$, then $N(x) \,\dot\cup\, xN(y) \subseteq N(x) \,\dot\cup\, xN(w)$ if and only if $N(y) \subseteq N(w)$. For (d), the first two conditions give $N(x) \cap N(v) = \emptyset$ (by (2.2.2)) and

$$N(v) \subseteq N(xy) = N(x) + xN(y) \subseteq N(x) \cup xN(y),$$

so $N(v) \subseteq xN(y)$. The third condition gives by (2.2.2) again that $N(y) \cap N(w) = \emptyset$, hence $xN(y) \cap xN(w) = \emptyset$. Therefore, $N(v) \cap xN(w) = \emptyset$. This implies that

$$N(v) \cap N(xw) = N(v) \cap (N(x) + xN(w)) = (N(v) \cap N(x)) + (N(v) \cap xN(w)) = \emptyset$$

which gives the conclusion of (d) by (2.2.2). The simple proof of (e) is omitted. □

2.5. **Faithful protorootoids.** The next Lemma follows immediately from the definitions and the cocycle property.

Lemma. *The following conditions* (i)–(iv) *on a protorootoid* (G, Λ, N) *are equivalent:*

(i) *For all* $a \in \mathrm{ob}(G)$ *and* $g \in {}_aG$, *one has* $N(g) = 0$ *only if* $g = 1_a$.
(ii) *For all* $a \in \mathrm{ob}(G)$ *and* $g \in {}_aG$, *one has* $g \,{}_a{\le}\, 1_a$ *only if* $g = 1_a$.
(iii) *For all* $a \in \mathrm{ob}(G)$ *and any* $g, h \in {}_aG$ *with* $N(g) = N(h)$, *one has* $g = h$.
(iv) *The weak right preorders of* $\mathscr{R}$ *are all partial orders.*

If these conditions hold, then $\mathscr{R}$ is said to be a *faithful* protorootoid. In that case, there is an order isomorphism $x \mapsto N(x) \colon {}_aG \xrightarrow{\cong} {}_a\mathscr{L}$ and the weak right preorder ${}_a{\le}$ on ${}_aG$ will be called the *weak right order* (of $\mathscr{R}$, or G, at a).

2.6. **Compatible expressions.** If G is a category, an expression e in G is defined to be a diagram

$$a_0 \xleftarrow{g_1} a_1 \leftarrow \cdots \leftarrow a_{n-1} \xleftarrow{g_n} a_n \tag{2.6.1}$$

of objects a_i and morphisms g_i of G, where $n \in \mathbb{N}$. This expression may be denoted more compactly as $e = {}_{a_0}[g_1, \ldots, g_n]_{a_n}$. Its value g is defined as $g := g_1 \ldots g_n \in \mathrm{mor}(G)$ if $n > 0$ and as 1_{a_0} if $n = 0$, and its length is defined to be n. By abuse of terminology and notation, we may simply say that $g_1 \cdots g_n$ is an expression (with value g) or that $g = g_1 \cdots g_n$ is an expression.

Now assume that G is the underlying groupoid of a protorootoid $\mathscr{R} = (G, \Lambda, N)$. For the above expression e, the cocycle condition implies by induction on n that

$$N(g) = N(g_1) + g_1 N(g_2) + \ldots + g_1 \cdots g_{n-1} N(g_n). \tag{2.6.2}$$

The expression e is said to be *compatible* if $g_1 \cdots g_{i-1} N(g_i) \cap g_1 \cdots g_{j-1} N(g_j) = \emptyset$ for all $i \neq j$, or equivalently, if

$$N(g) = N(g_1) \,\dot\cup\, g_1 N(g_2) \,\dot\cup\, \ldots \,\dot\cup\, g_1 \cdots g_{n-1} N(g_n). \tag{2.6.3}$$

One easily sees by induction on n (see (2.2.2) for $n = 2$) that e is compatible if and only if

$$\emptyset \subseteq N(g_1) \subseteq N(g_1 g_2) \subseteq \ldots \subseteq N(g_1 \cdots g_n) \tag{2.6.4}$$

in ${}_{a_0}\mathscr{L}$ or equivalently if and only if

$$1_{a_0} \leq g_1 \leq g_1 g_2 \leq \ldots \leq g_1 \cdots g_n \tag{2.6.5}$$

in ${}_{a_0}G$.

The following *substitution property* of compatible expressions partly reduces the study of compatible expressions to the study of those of length two. Its simple proof is omitted.

Lemma. *Let e be an expression as above and let $0 = i_0 \leq i_1 \leq \ldots \leq i_p = n$ be integers. For $j = 1, \ldots, p$, let e_j denote the expression ${}_{a_{i_{j-1}}}[g_{i_{j-1}+1}, \ldots, g_{i_j}]_{a_{i_j}}$ and h_j denote the value of e_j. Then e is compatible if and only if $e_1, \ldots, e_p$ and ${}_{a_0}[h_1, \ldots, h_p]_{a_n}$ are all compatible expressions.*

Remarks. Equipped with suitable face and degeneracy operations, by composition and insertion of identity morphisms, the expressions in G naturally determine a simplicial set (see [18]) known as the nerve of G, which is involved in the construction of the classifying space BG of the category G. The compatible expressions constitute a simplicial subset of the nerve.

2.7. **Orthogonality of morphisms.** Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid. Two morphisms g, h of G are said to be *orthogonal* if they have a common codomain a and $N(g) \cap N(h) = \emptyset$ in ${}_a\Lambda$.

The following trivial lemma records the way in which each of the three concepts of weak preorders, compatibility of expressions and orthogonality of morphisms can be expressed in terms of each of the others. The proof is omitted.

Lemma. *Let $x, y \in \mathrm{mor}(G)$ with $\exists xy$. Then the following conditions are equivalent:*

(i) *x^* and y are orthogonal morphisms.*
(ii) *xy is a compatible expression.*
(iii) *$x \leq xy$ in big weak preorder.*

2.8. **Protomeshes.** Call a pair (R, L) consisting of a Boolean ring R and a subset L of R a *protomesh.* Regard L as a poset with order induced by that of R. A morphism $\theta\colon (R, L) \to (R', L')$ of protomeshes is defined to be a ring homomorphism $\theta\colon R \to R'$ such that $\theta(A) \in L'$ for all $A \in L$. This defines a category of protomeshes, with its composition given by composition of ring homomorphisms. The morphism θ of protomeshes induces a morphism $L \to L'$ in **Ord**. For a protomesh (R, L) and $\Gamma \in R$, define a protomesh $(R, \Gamma + L)$ where $\Gamma + L := \{\, \Gamma + \Delta \mid \Delta \in L \,\}$.

The protomesh (R, L) is said to be *standard* if $0 \in L$. If (R, L) is any protomesh, then $(R, \Gamma + L)$ is a standard protomesh for each $\Gamma \in L$.

2.9. **Relation between weak orders.** The following proposition describes relationships between the weak orders of a protorootoid.

Proposition. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid with big weak order $\mathscr{L}$.*
(a) *Let $a, b \in \mathrm{ob}(G)$ and $g \in {}_bG_a$. Then $\Lambda(g)\colon {}_a\Lambda \to {}_b\Lambda$ is an isomorphism of protomeshes $({}_a\Lambda, {}_a\mathscr{L}) \xrightarrow{\cong} ({}_b\Lambda, \Gamma + {}_b\mathscr{L})$ where $\Gamma := N(g) \in {}_b\mathscr{L}$.*
(b) *For fixed $b \in \mathrm{ob}(G)$, the protomesh $({}_b\Lambda, {}_b\mathscr{L})$ completely determines the family of isomorphism types of the protomeshes $({}_a\Lambda, {}_a\mathscr{L})$ for $a \in \mathrm{ob}(G[b])$.*

Proof. For all $x \in {}_aG$, $(\Lambda(g))(N_x) = N(g) + N(gx) = \Gamma + N(gx)$. Since the map $x \mapsto gx\colon {}_aG \to {}_bG$ is bijective, ${}_b\mathscr{L} = \{\, N(gx) \mid x \in {}_aG \,\}$ and (a) follows. Then (b) holds since the isomorphism types of protomeshes $({}_a\Lambda, {}_a\mathscr{L})$ for $a \in \mathrm{ob}(G[b])$ coincide with the isomorphism types of protomeshes $({}_b\Lambda, \Gamma + {}_b\mathscr{L})$ for $\Gamma \in {}_b\mathscr{L}$, by (a). $\square$

2.10. **Categories of protorootoids.** By a *morphism of protorootoids*, we shall mean a morphism in the category **Prd** defined below, unless otherwise specified.

Definition. The category **Prd** is the category with protorootoids as objects and in which a morphism $(G, \Lambda, N) \to (G', \Lambda', N')$ is a pair (α, μ) such that
(i) $\alpha\colon G \to G'$ is a groupoid homomorphism i.e. a functor.
(ii) $\mu\colon \Lambda \to \Lambda'\alpha$ is a natural transformation (between functors $G \to \mathbf{BRng}$).
(iii) $\mu N = N'\alpha$ i.e. for all $a \in \mathrm{ob}(G)$ and $g \in {}_aG$, $\mu_a(N_g) = N'_{\alpha(g)}$ in $\Lambda'(\alpha(a))$.

For another morphism $(\alpha', \mu')\colon (G', \Lambda', N') \to (G'', \Lambda'', N'')$, the composite morphism $(\alpha', \mu')(\alpha, \mu)\colon (G, \Lambda, N) \to (G'', \Lambda'', N'')$ is defined to be the pair $(\alpha'\alpha, (\mu'\alpha)\mu)$ where the composite natural transformation $(\mu'\alpha)\mu\colon \Lambda \to \Lambda''\alpha'\alpha$ has component at $a \in \mathrm{ob}(G)$ given by $((\mu'\alpha)\mu)_a := \mu'_{\alpha(a)}\mu_a$.

The full subcategory of **Prd** consisting of faithful protorootoids is denoted **FPrd**. For a fixed groupoid G, the category G-**Prd** of *G-protorootoids* is defined as the subcategory of **Prd** with only those objects of the form (G, Λ, N) and with only those morphisms of the form (Id_G, μ). The category $\mathbf{Prd}_1$ of *unitary protorootoids* is the (not full) subcategory of **Prd** with unitary protorootoids as objects and morphisms

(α, ν) in **Prd** between unitary protorootoids such that each component of ν is a morphism in **BAlg** (i.e. the components are *unital* ring homomorphisms).

Lemma. *For any morphism* $(\alpha, \mu): (G, \Lambda, N) \to (G', \Lambda', N')$ *of protorootoids, the map induced by* α *on morphisms restricts for each* $a \in \mathrm{ob}(G)$ *to a weak preorder preserving map* ${}_a\alpha: ({}_aG,\ {}_a{\leq}) \to ({}_{a'}G',\ {}_{a'}{\leq})$ *where* $a' := \alpha(a)$*. Further,* $\mu_a: ({}_a\Lambda, {}_a\mathscr{L}) \to ({}_{\alpha(a)}\Lambda', {}_{\alpha(a)}\mathscr{L}')$ *is a morphism of protomeshes, where* $\mathscr{L}'$ *is the big weak order of* (G', Λ', N')*.*

Proof. Let $g_1, g_2 \in {}_aG$ with $g_1\ {}_a{\leq}\ g_2$ i.e. $N(g_1) \subseteq N(g_2)$. Since $\mu_a: {}_a\Lambda \to {}_{a'}\Lambda'$ is a homomorphism of Boolean rings, it is order preserving, and the definitions give

$$N'(\alpha(g_1)) = \mu_a(N(g_1)) \subseteq \mu_a(N(g_2)) = N'(\alpha(g_2))$$

i.e. $\alpha(g_1)\ {}_{a'}{\leq}\ \alpha(g_2)$. Both assertions follow. $\square$

2.11. **Inverse image.** Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid and $i: H \to G$ be a groupoid morphism. Define the *inverse image protorootoid* $i^\natural\mathscr{R} := (H, \Lambda i, Ni)$. There is a morphism $i^\flat = (i, \mathrm{Id}_{\Lambda i}): i^\natural\mathscr{R} \to \mathscr{R}$ in **Prd**.

Note that $i^\natural$ becomes a functor $i^\natural: G\text{-}\mathbf{Prd} \to H\text{-}\mathbf{Prd}$ if one defines $i^\natural(\mathrm{Id}_G, \nu) = (\mathrm{Id}_H, \nu i)$ for any morphism (Id_G, ν) of G-protorootoids.

Also, $i^\flat$ has the following universal property: given a protorootoid $\mathscr{R}' = (H, \Lambda', N')$ and a morphism $f = (i, \mu): \mathscr{R}' \to \mathscr{R}$ in **Prd**, there is a unique morphism $g: \mathscr{R}' \to i^\natural(\mathscr{R})$ in H-**Prd** such that $f = i^\flat g$ in **Prd** (namely, $g = (\mathrm{Id}_H, \mu)$).

2.12. **Restriction.** If $i: H \to G$ is the inclusion morphism of a subgroupoid H into G, then $i^\natural\mathscr{R}$ is called the *restriction* of $\mathscr{R}$ to H and is denoted sometimes as $\mathscr{R}_H := i^\natural\mathscr{R}$.

2.13. **Coverings.** A morphism $f = (\alpha, \nu): \mathscr{R}' \to \mathscr{R}$ in **Prd** is said to be a *covering morphism* or *covering* if α is a covering morphism of groupoids and ν is a natural isomorphism. Equivalently, f is a covering if α is a covering of groupoids and the natural morphism $\mathscr{R}' \to \alpha^\natural(\mathscr{R})$ is an isomorphism in **Prd**. In that case, $\mathscr{R}'$ is called a covering protorootoid or *covering* of $\mathscr{R}$. If the groupoid morphism α is a covering quotient morphism, then $\mathscr{R}$ is called a *covering quotient* of $\mathscr{R}'$ and f is called a *covering quotient morphism.* A *universal covering* of a protorootoid $\mathscr{R}$ is a covering $(\alpha, \nu): \mathscr{R}' \to \mathscr{R}$ such that α is a universal covering in **Gpd** of the underlying groupoid of $\mathscr{R}$. Such a universal covering exists and it is unique up to isomorphism as an object of $\mathbf{Prd}/\mathscr{R}$.

2.14. Another category $\mathbf{Prd}'$ of protorootoids, which will only be considered occasionally, is defined as in 2.10(i)–(iii) but taking $\mu: \Lambda'\alpha \to \Lambda$ in (ii) and replacing (iii) by the following: $N = \mu N'\alpha$ i.e. for $a \in \mathrm{ob}(G)$ and $g \in {}_aG$, one has $N_g = \mu_a(N'_{\alpha(g)})$. For another morphism $(\alpha', \mu'): (G', \Lambda', N') \to (G'', \Lambda'', N'')$, the composite $(\alpha', \mu')(\alpha, \mu): (G, \Lambda, N) \to (G'', \Lambda'', N'')$ in $\mathbf{Prd}'$ is the pair $(\alpha'\alpha, \mu(\mu'\alpha))$ where $\mu(\mu'\alpha): \Lambda''\alpha'\alpha \to \Lambda$ has component at $a \in \mathrm{ob}(G)$ given by $(\mu(\mu'\alpha))_a := \mu_a\mu'_{\alpha(a)}$.

Define G-$\mathbf{Prd}'$ from $\mathbf{Prd}'$ in a similar way as G-**Prd** is defined from **Prd**.

2.15. **Groupoid-preorders.** The category **Gpd-PreOrd** of groupoid-preorders is defined as follows. It has as objects groupoids G such that for each $a \in \mathrm{ob}(G)$, there is a given preorder ${}_a\!\leq$ on ${}_aG$, called the *weak right preorder* of G at a. A morphism $G \to H$ in **Gpd-PreOrd** is a groupoid homomorphism $\theta \colon H \to G$ such that for each $a \in \mathrm{ob}(H)$, the restriction ${}_a\theta$ of θ to a function ${}_aG \to {}_{\theta(a)}H$ is a morphism in **PreOrd**. Composition in **Gpd-PreOrd** is given by composition of underlying groupoid morphisms. The full subcategory of **Gpd-PreOrd** consisting of groupoids for which all the weak right preorders are partial orders is called the category of groupoid-orders.

Formally, a groupoid-preorder is a pair $(G, \leq)$ consisting of a groupoid G and a preorder $\leq$, which is called the *big weak right preorder*, on $\mathrm{mor}(G) = \dot\bigcup_{a \in \mathrm{ob}(G)} {}_aG$ such that $\leq$ restricts to the weak right preorder of ${}_aG$, and elements in different stars ${}_aG$ of G are incomparable.

There is a natural forgetful functor $\mathfrak{P} \colon \mathbf{Prd} \to \mathbf{Gpd\text{-}PreOrd}$ which on objects takes a protorootoid $\mathscr{R} = (G, \Lambda, N)$ to the groupoid G endowed with the collection of weak right preorders of $\mathscr{R}$, and which takes a morphism of protorootoids to the underlying morphism of groupoids (which is a morphism in **Gpd-PreOrd** by Lemma 2.10). The full subcategory of **Gpd-PreOrd** with objects the groupoid-preorders which are isomorphic to $\mathfrak{P}(\mathscr{R})$ for some protorootoid $\mathscr{R}$ is denoted $\mathbf{Gpd\text{-}PreOrd}_P$ and called the category of *protorootoidal groupoid-preorders*. In this definition, "isomorphic" could be replaced by "equal" since if $i \colon (G, \leq) \to \mathfrak{P}(\mathscr{R})$ is an isomorphism in **Gpd-PreOrd**, then (regarding i just as a morphism of groupoids) $\mathfrak{P}(i^\natural(\mathscr{R})) = (G, \leq)$.

2.16. **Order isomorphism.** A *preorder isomorphism* from a protorootoid $\mathscr{R}$ to a protorootoid $\mathscr{T}$ is by definition an isomorphism $\mathfrak{P}(\mathscr{R}) \to \mathfrak{P}(\mathscr{T})$ in **Gpd-PreOrd** i.e. an isomorphism $\theta \colon G \to H$ from the underlying groupoid of $\mathscr{R}$ to that of $\mathscr{T}$ such that the induced maps ${}_a\theta \colon {}_aG \to {}_{\theta(a)}(H)$ for $a \in \mathrm{ob}(G)$ are all preorder isomorphisms in the corresponding right weak preorders. Similarly, one defines *order isomorphisms* of faithful protorootoids.

Remarks. (1) Many, though not all, properties of (and definitions concerning) protorootoids $\mathscr{R}$ may be expressed completely in terms of $\mathfrak{P}(\mathscr{R})$. For example, faithfulness of a protorootoid is such a property, and the simplicial set of compatible expressions depends up to isomorphism only on the preorder isomorphism type.

(2) Subsequent papers will give characterizations of protorootoidal groupoid-preorders and show how an analogue of part of the theory of protorootoids and rootoids may be developed in the context of $\mathbf{Gpd\text{-}PreOrd}_P$.

3. Principal protorootoids

3.1. **Groupoid generators.** Let G be a groupoid and $S \subseteq \mathrm{mor}(G)$. The subgroupoid H of G generated by S is defined to the subgroupoid of G containing all identity morphisms of G and all morphisms g of G which are expressible as a product $s_1 \cdots s_n$ with $s_i \in S \cup S^*$. One says more briefly that S *generates* H. If S

generates G, a corresponding *length function* $l_S\colon \operatorname{mor}(G) \to \mathbb{N}$ is defined by

$$l_S(g) := \min(\{\, n \in \mathbb{N} \mid g = s_1 \cdots s_n, s_i \in S \cup S^* \,\}) \tag{3.1.1}$$

if g is not an identity morphism, and $l_S(g) := 0$ if g is an identity morphism.

3.2. **Rank in Boolean rings.** A finite Boolean ring B is a Boolean algebra since the join of all (finitely many) of its elements is a maximal element of B and hence an identity element of B. Recall that a finite Boolean algebra B is isomorphic to the Boolean algebra of subsets of a finite set of uniquely determined cardinality $\operatorname{rank}(B)$ (equal to the number of atoms of B).

Now let B denote an arbitrary Boolean ring. For $x \in B$, the principal ideal generated by x is

$$xB = x \cap B = \{\, x \cap y \mid y \in B \,\} = \{\, x' \in B \mid x' \leq x \,\},$$

which, regarded as a subring of B, is itself a Boolean ring. If xB is finite, say that x is of finite *rank* $\operatorname{rank}(x) := \operatorname{rank}(xB)$. The atoms of B are its elements of rank 1.

Though the following is well known, a proof is given for completeness.

Lemma. *Let B be a Boolean ring, and U be the set of atoms of B. Let $\wp'(U)$ be the (possibly non-unital) subring of $\wp(U)$ with the finite subsets of U as its elements.*

(a) *If $x, y \in B$ are of finite rank, then so are $x \cup y$ and $x \cap y$, and* $\operatorname{rank}(x) + \operatorname{rank}(y) = \operatorname{rank}(x \cap y) + \operatorname{rank}(x \cup y)$.

(b) *Let B' be the subring (also an ideal and order ideal) of B consisting of elements of finite rank. Then the map $\theta\colon x \mapsto \{\, y \leq x \mid \operatorname{rank}(y) = 1 \,\}$ defines an isomorphism of Boolean rings $\theta\colon B' \to \wp'(U)$.*

(c) *If $x, y \in B'$, then $\theta(x \cup y) + \theta(x \cap y) = \theta(x) + \theta(y)$.*

Proof. If $x, y \in B$ satisfy $x \cap y = \emptyset$, then x and y are orthogonal idempotents so $(x + y)B \cong xB \times yB$ as ring. If x and y are also of finite rank, this implies that $\operatorname{rank}(x+y) = \operatorname{rank}(x)+\operatorname{rank}(y)$, so $x+y$ is of finite rank, and $\theta(x+y) = \theta(x) \,\dot\cup\, \theta(y)$. Now let $x, y \in B$ be arbitrary elements of finite rank. Since $x \cap y$ and $x + (x \cap y)$ are orthogonal idempotents (of finite rank since they are in $[\emptyset, x]_B$), one has

$$\operatorname{rank}(x + (x \cap y)) + \operatorname{rank}(x \cap y) = \operatorname{rank}(x)$$

and $\theta(x + (x \cap y)) \,\dot\cup\, \theta(x \cap y) = \theta(x)$. But y and $x + (x \cap y)$ are also orthogonal, with sum $x \cup y$, so

$$\operatorname{rank}(x + (x \cap y)) + \operatorname{rank}(y) = \operatorname{rank}(x \cup y)$$

and $\theta(x + (x \cap y)) \,\dot\cup\, \theta(y) = \theta(x \cup y)$. These formulae easily imply (a), (c) and that $\theta(x \cup y) = \theta(x) \cup \theta(y)$. It is also clear by the definition of θ that $\theta(x \cap y) = \theta(x) \cap \theta(y)$. Using $x \cup y = (x + y) \,\dot\cup\, (x \cap y)$ and an analogous fact in $\wp'(U)$, it follows that

$$\begin{aligned}(\theta(x) + \theta(y)) \,\dot\cup\, (\theta(x) \cap \theta(y)) &= \theta(x) \cup \theta(y) \\ &= \theta(x \cup y) = \theta(x + y) \,\dot\cup\, \theta(x \cap y) = \theta(x + y) \,\dot\cup\, (\theta(x) \cap \theta(y))\end{aligned}$$

from which $\theta(x + y) = \theta(x) + \theta(y)$. Hence $\theta\colon B' \to \wp'(U)$ is a ring homomorphism. One readily checks that an inverse function $\theta^{-1}\colon \wp'(U) \to B'$ is given by $X \mapsto \cup_{x \in X} x$ (the join in B of X) for finite $X \subseteq U$. $\square$

3.3. **Terminology for protorootoids.** The following definition collects the basic terminology used in these papers for protorootoids. Complete and principal protorootoids are the two most important classes; others are technically useful in relation to them or in formulating results in their natural generality.

Definition. Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid with big weak order $\mathscr{L}$.

(a) $\mathscr{R}$ is said to be *connected* (resp., *simply connected*) if its underlying groupoid G is connected (resp., simply connected).
(b) $\mathscr{R}$ is *complemented* if it is unitary and for each $a \in \mathrm{ob}(G)$ and $A \in {}_a\mathscr{L}$, one has $A^{\complement} := 1_{\Lambda(a)} + A \in {}_a\mathscr{L}$.
(c) $\mathscr{R}$ is *complete* if for each $a \in {}_aG$, ${}_a\mathscr{L}$ is a complete lattice.
(d) $\mathscr{R}$ is *interval finite* if for each $a \in \mathrm{ob}(G)$ and each morphism $g \in {}_aG$, the interval $[\emptyset, N(g)]_{{}_a\mathscr{L}} := \{ A \in {}_a\mathscr{L} \mid A \subseteq N(g) \}$ in ${}_a\mathscr{L}$ is finite.
(e) $\mathscr{R}$ is *cocycle finite* if for each $a \in \mathrm{ob}(G)$ and each morphism $g \in {}_aG$, the element $N(g)$ is of finite rank in ${}_a\Lambda$. In that case, define a function $l_N \colon \mathrm{mor}(G) \to \mathbb{N}$ by $l_N(g) := \mathrm{rank}(N(g))$.
(f) An element $s \in \mathrm{mor}(G)$, say $s \in {}_aG_b$, is an *atomic morphism* of $\mathscr{R}$ (or G) if $N(s)$ is an atom of the weak order ${}_a\mathscr{L}$. Let $A_{\mathscr{R}}$ denote the set of atomic morphisms of $\mathscr{R}$.
(g) An element $s \in \mathrm{mor}(G)$, say $s \in {}_aG_b$, is a *simple morphism* of $\mathscr{R}$ (or of G) if $N(s)$ is an atom of the Boolean ring ${}_a\Lambda$. Let $S_{\mathscr{R}}$ denote the set of simple morphisms of $\mathscr{R}$.
(h) $\mathscr{R}$ is *atomically generated* (resp., *simply generated*) if $A_{\mathscr{R}}$ (resp., $S_{\mathscr{R}}$) generates G.
(i) $\mathscr{R}$ is *principal* if it is cocycle finite, simply generated and $l_S = l_N \colon \mathrm{mor}(G) \to \mathbb{N}$ where $S := S_{\mathscr{R}}$.
(j) $\mathscr{R}$ is *preprincipal* if it is faithful and interval finite, and for all $a \in \mathrm{ob}(G)$, $g \in {}_aG$ and $s \in {}_aA$ (where $A := A_{\mathscr{R}}$), either $N(g) \cap N(s) = \emptyset$ or $N(s) \subseteq N(g)$ (i.e. either s^*g or sh is a compatible expression, where $h := s^*g$).
(k) $\mathscr{R}$ is *abridged* if for each $a \in \mathrm{ob}(G)$, ${}_a\Lambda$ is generated as Boolean ring by ${}_a\mathscr{L}$.
(l) $\mathscr{R}$ is *saturated* if for every $a \in \mathrm{ob}(G)$ and every $g \in {}_aG$, every maximal totally ordered subset of $[\emptyset, N(g)]_{{}_a\mathscr{L}}$ is also a maximal totally ordered subset of $[\emptyset, N(g)]_{{}_a\Lambda}$.
(m) $\mathscr{R}$ is *pseudoprincipal* if for every $a \in \mathrm{ob}(G)$ and $h, g \in {}_aG$ with $N(h) \neq \emptyset$, there exists $x \in {}_aG$ with $\emptyset \neq N(x) \subseteq N(h)$ and either $N(x) \subseteq N(g)$ or $N(x) \cap N(g) = \emptyset$.
(n) $\mathscr{R}$ is *regular* if for every $a \in \mathrm{ob}(G)$ and every non-empty directed subset X of ${}_a\mathscr{L}$ with a join x in ${}_a\mathscr{L}$, x is also the join of X as a subset of ${}_a\Lambda$.

The set S of simple morphisms of a simply generated protorootoid $\mathscr{R} = (G, \Lambda, N)$ is called the set of *simple generators* of $\mathscr{R}$ (or less precisely, of G). Note that $S_{\mathscr{R}}$ and $A_{\mathscr{R}}$ may be empty for an arbitrary protorootoid $\mathscr{R}$.

The property of being connected (resp., simply connected, complete, interval finite, atomically generated, preprincipal, pseudoprincipal) depends only on the preorder isomorphism type of the protorootoid

3.4. Basic properties of atomic and simple morphisms are listed below.

Lemma. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid. Set $A := A_{\mathscr{R}}$ and $S := S_{\mathscr{R}}$. Then:*
 (a) *$A = A^*$ and $S = S^*$.*
 (b) *$S \subseteq A$.*
 (c) *A (and therefore S) contains no identity morphism of G.*

Proof. To show $S = S^*$ in (a), observe that for any $g \in {}_bG_a$, $\Lambda(g)$ maps the set of atoms of ${}_a\Lambda$ to the set of atoms of ${}_b\Lambda$. If $s \in {}_aS_b$, then $N(s)$ is an atom of ${}_a\Lambda$ and so $N(s^*) = s^*N(s)$ is an atom of ${}_b\Lambda$ i.e. $s^* \in {}_bS$. This shows that $S^* \subseteq S$. Hence $S = S^{**} \subseteq S^* \subseteq S$. To prove that $A = A^*$, it will suffice to show that if $s \in \mathrm{mor}(G)$ is not in A, then $s^* \notin A$ also. Note that either $N(s) = \emptyset$ or there exist $x, y \in \mathrm{mor}(G)$ with $s = xy$ and $\emptyset \subsetneq N(x) \subsetneq N(s)$. In the first case $N(s^*) = s^*N(s) = \emptyset$ and in the second case, $s^* = y^*x^*$ with $\emptyset \subsetneq N(y^*) \subsetneq N(s^*)$ (see (2.2.2)). In either case, $s^* \notin A$. This proves (a). Parts (b)–(c) are immediate consequences of the definitions. □

3.5. The following is a very useful property of interval finite protorootoids.

Lemma. *If the protorootoid $\mathscr{R} = (G, \Lambda, N)$ is interval finite and faithful, then G is atomically generated.*

Proof. Define a function $L\colon \mathrm{mor}(G) \to \mathbb{N}$ by $L(g) = |[\emptyset, N(g)]_{{}_a\mathscr{L}}|$ for $g \in {}_aG$. We show that $g \in \mathrm{mor}(G)$ is in the subgroupoid G' of G generated by $A := A_{\mathscr{R}}$ by induction on $L(g)$. If $L(g) = 0$, then $N(g) = \emptyset$ and $g = 1_a \in G'$ since $\mathscr{R}$ is faithful. Suppose $L(g) > 0$. There is $s \in {}_aA$ with $N(s) \subseteq N(g)$. Set $g' := s^*g \in {}_bG$. Since $N(s) \subseteq N(g)$, it follows that $N(s^*) \cap N(g') = \emptyset$. Using Proposition 2.4, one checks that the map $x \mapsto x' := sx$ defines a bijection

$$\{\, x \in {}_bG \mid 1_b \;{}_b{\leq}\; x \;{}_b{\leq}\; g' \,\} \xrightarrow{\cong} \{\, x' \in {}_aG \mid s \;{}_a{\leq}\; x' \;{}_a{\leq}\; g \,\}.$$

Since $1_a \;{}_a{\leq}\; g$ but $s \;{}_a{\not\leq}\; 1_a$, it follows that $L(g') < L(g)$. By induction, $g' \in \mathrm{mor}(G')$ so $g = sg' \in \mathrm{mor}(G')$ as required. □

Remarks. If $\mathscr{R}$ is interval finite and faithful and H is a subgroupoid of G, then the restriction $\mathscr{R}_H$ is also interval finite and faithful, so the atomic morphisms of $\mathscr{R}_H$ form a set of generators of H.

3.6. Part (c) of the lemma below eliminates some redundancies from the definition of principal protorootoids.

Lemma. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid, $A := A_{\mathscr{R}}$ and $S := S_{\mathscr{R}}$.*
 (a) *If $\mathscr{R}$ is cocycle finite, it is interval finite.*
 (b) *If $\mathscr{R}$ is simply generated, it is cocycle finite and atomically generated, and for all $g \in \mathrm{mor}(G)$, $l_N(g) \leq l_S(g)$.*
 (c) *$\mathscr{R}$ is principal if and only if it is simply generated and for all $g \in \mathrm{mor}(G)$, $l_S(g) \leq l_N(g)$.*
 (d) *If $\mathscr{R}$ is interval finite, it is regular.*

Proof. Part (a) holds since for $g \in \mathrm{mor}(g)$,

$$[\emptyset, N_g]_{{}_a\mathscr{L}} \subseteq [\emptyset, N_g]_{{}_a\Lambda}. \tag{3.6.1}$$

For (b), assume that $\mathscr{R}$ is simply generated. By Lemma 3.4(b), $\mathscr{R}$ is atomically generated. Let $g \in \mathrm{mor}(G)$, say $g = s_1 \cdots s_n$ where $s_i \in S$ and $n = l_S(g)$. By the cocycle condition,

$$N_g = \sum_{i=1}^{n} s_1 \dots s_{i-1}(N_{s_i}).$$

In this, $\mathrm{rank}(s_1 \cdots s_{i-1}(N_{s_i})) = \mathrm{rank}(N_{s_i}) = 1$, so by Lemma 3.2(a), $\mathrm{rank}(N_g) \leq \sum_{i=1}^{n} 1 = n = l_S(g)$. This completes the proof of (b), and (c) follows immediately from (b) and the definition of principal protorootoids. Part (d) holds since if a nonempty subset X of ${}_a\mathscr{L}$ has a join x in ${}_a\mathscr{L}$, then X is finite and x is the maximum element of X, so x is also the join of X in ${}_a\Lambda$. $\square$

3.7. Principal protorootoids may also be characterized as follows.

Lemma. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid, $A := A_{\mathscr{R}}$ and $S := S_{\mathscr{R}}$. Then $\mathscr{R}$ is principal if and only if it is faithful and atomically generated and $A \subseteq S$. In that case, $A = S$.*

Proof. Suppose that $\mathscr{R}$ is faithful, atomically generated and that $A \subseteq S$. Then $A = S$ by Lemma 3.4(b), so $\mathscr{R}$ is simply generated. Let $g \in {}_bG_a$ and $s \in {}_aS$ for some $a \in \mathrm{ob}(G)$. Then $N(gs) = N(g) + gN(s)$. Since $N(s)$ is an atom of ${}_a\Lambda$, $gN(s)$ is an atom of ${}_b\Lambda$. By Lemma 3.2, it follows that $l_N(gs) = l_N(g) + 1$ if $gN(s) \cap N(g) = \emptyset$, while otherwise, $gN(s) \subseteq N(g)$ and $l_N(gs) = l_N(g) - 1$. In particular, $l_N(gs) \in \{l_N(g) \pm 1\}$. This implies by induction on $l_S(g)$ that $l_S(g) \equiv l_N(g) \pmod 2$, and that $l_S(gs) \in \{l_S(g) \pm 1\}$ for $g \in \mathrm{mor}(G)$ and $s \in S$ if $\exists gs$. Since $l_S(x) = l_S(x^*)$ for all $x \in \mathrm{mor}(G)$, it follows that for $s \in S$, $g \in G$ with $\exists sg$, one has $l(sg) \in \{l(g) \pm 1\}$.

To show $\mathscr{R}$ is principal, it remains to show that for all $g \in \mathrm{mor}(G)$, $l_S(g) = l_N(g)$. This is proved by induction on $n := l_N(g)$. If $n = 0$, then $N(g) = \emptyset$, g is an identity morphism since $\mathscr{R}$ is faithful, and so $l_S(g) = 0 = n$. Next, suppose inductively that $l_S(g) = l_N(g)$ for all g with $l_N(g) < n$, where $n > 0$. Let $g \in {}_aG$ with $l_N(g) = n$. Since $n > 0$, $N(g) \neq \emptyset$. Since $\mathscr{R}$ is interval finite, the interval $[0, N(g)]_{{}_a\mathscr{L}}$ has an atom i.e. there is some $r \in {}_aA$ with $N(r) \subseteq N(g)$. Set $s = r^* \in S$. The cocycle condition implies $N(sg) = s(N(g) + N(r))$ and so $l_N(sg) = l_N(g) - 1 = n - 1$. By induction, $l_S(sg) = n - 1$. By Lemma 3.6(b),

$$n = l_N(g) \leq l_S(g) \in \{l_S(sg) \pm 1\} = \{(n-1) \pm 1\}$$

and thus $l_S(g) = n = l_N(g)$ as required.

Conversely, suppose that $\mathscr{R}$ is principal. If $g \in {}_aG$ with $N(g) = \emptyset$, then $l_N(g) = 0 = l_S(g)$ so $g = 1_a$. Hence $\mathscr{R}$ is faithful. Since $\mathscr{R}$ is simply generated, it is atomically generated by Lemma 3.6(b). Let $s \in {}_aA$ with $l_N(s) = l_S(s) = n$. Note $n > 0$ by Lemma 3.4(c). Write $s = s_1 \cdots s_n$ with $s_i \in S$. Then $l_N(s_1 \cdots s_i) = l_S(s_1, \cdots s_i) = i$ for $i = 0, \dots, n$ which implies that $\emptyset \subsetneq N(s_1) \subsetneq N(s_1 s_2) \subsetneq \dots \subsetneq N(s)$. In particular, since $N(s)$ is an atom of ${}_a\mathscr{L}$, it follows that $n = 1$ and $s = s_1 \in S$. This shows that $A \subseteq S$. One has $S = A$ by Lemma 3.4(b), completing the proof. $\square$

3.8. Let $\{\pm 1\}$ be the group of units of the ring $\mathbb{Z}$ and regarded it as a groupoid with one object. Let G be a groupoid and S be a set of generators of G. A sign character of (G,S) is defined to be a groupoid homomorphism $\epsilon\colon G \to \{\pm 1\}$ such that $\epsilon(s) = -1$ for all $s \in S$. If it exists, it is unique, since it is given on morphisms by $\epsilon(g) = (-1)^{l_S(g)}$.

Corollary. *Let $\mathscr{R} = (G, \Lambda, N)$ be a principal protorootoid and $S := S_{\mathscr{R}}$. Then (G,S) admits a sign character.*

Proof. The preceding proof shows that if $g \in \mathrm{mor}(G)$ and $s \in S$ and $l_S(gs) = l_S(g) \pm 1$ if $\exists gs$. Also, $l_S(s) = 1$ by Lemma 3.4(c). Thus, one may take $\epsilon(g) := (-1)^{l_S(g)}$. □

3.9. **Abridgement.** Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid. Define Λ' as the G-subrepresentation (in **BRng**) of Λ generated by the elements ${}_aN(g) \in {}_a\Lambda$ for $a \in \mathrm{ob}(G)$ and $g \in {}_aG$. As functor, Λ' is determined on objects by defining, for $a \in \mathrm{ob}(G)$, $\Lambda'(a)$ as the subring of ${}_a\Lambda$ generated by all elements ${}_aN(g) \in {}_a\Lambda$ with $g \in {}_aG$. For a morphism $g \in {}_aG_b$, $\Lambda(g)$ maps $\Lambda'(b)$ into $\Lambda'(a)$ (since for $h \in {}_bG$, $gN(h) = N(gh) + N(g)$) and so $\Lambda'(g)$ may be defined as the restriction of $\Lambda(g)$ to a morphism $\Lambda'(g)\colon \Lambda'(b) \to \Lambda'(a)$ in **BRng**. Obviously, N restricts to a G-cocycle N' for Λ', giving a protorootoid (G, Λ', N') which will be called the *abridgement* $\mathscr{R}^a := (G, \Lambda', N')$ of $\mathscr{R}$. The protorootoid $\mathscr{R}$ is abridged as defined in 3.3(k) if it is equal to $\mathscr{R}^a$.

It is immediate from the definition that a protorootoid $\mathscr{R}$ and its abridgement have equal underlying groupoid-preorders: $\mathfrak{P}(\mathscr{R}) = \mathfrak{P}(\mathscr{R}^a)$. The big weak orders of $\mathscr{R}$ and $\mathscr{R}^a$ are also equal as posets (though their corresponding protomeshes $({}_a\Lambda, {}_a\mathscr{L})$ and $({}_a\Lambda', {}_a\mathscr{L})$ differ). The set of atomic (resp., simple) morphisms of $\mathscr{R}$ is equal to (resp., a subset of) the set of atomic (resp., simple) morphisms of $\mathscr{R}^a$. Any property of protorootoids $\mathscr{R}$ which depends only on the preorder isomorphism type of $\mathscr{R}$ holds for $\mathscr{R}$ if and only if it holds for $\mathscr{R}^a$. If $\mathscr{R}$ is faithful, or has any one of the properties in 3.3(a)–(e) or (h)–(n), then $\mathscr{R}^a$ has that same property (using Lemma 3.7 for (i)); the converse holds for faithfulness, (a), (c), (d), atomically generated in (h), (j) and (m).

Remarks. The abridgement of a unitary protorootoid need not be unitary. However, there is an analogue of abridgement for unitary protorootoids. It attaches to a unitary protorootoid $\mathscr{R} = (G, \Lambda, N)$ a unitary protorootoid (G, Λ'', N'') where for $a \in \mathrm{ob}(G)$, $\Lambda'(a)$ is the subring of $\Lambda(a)$ generated by $\{1_{\Lambda(a)}\} \cup \{N(g) \mid g \in {}_aG\}$, and N'' is the evident restriction of N. One may have for example, that $\mathscr{R}^a = (G, \Lambda', N')$ where $\Lambda'(b)$ is the Boolean ring of all finite subsets of some set $U(b)$ whereas $\Lambda''(b)$ is the Boolean algebra of all subsets of $U(b)$ which are finite or cofinite (that is, have finite complement) in $U(b)$.

3.10. Let $\mathbf{Prd}^a$ denote the full subcategory of **Prd** consisting of abridged protorootoids. There is an abridgment functor $\mathfrak{A}\colon \mathbf{Prd} \to \mathbf{Prd}^a$ with $\mathfrak{A}(\mathscr{R}) = \mathscr{R}^a$ for $\mathscr{R}$ as above, and defined on morphisms as follows. Let $\mathscr{T} := (H, \Gamma, M)$ be a protorootoid with abridgement $\mathscr{T}^a = (H, \Gamma', M')$, and let $f = (\alpha, \nu)\colon \mathscr{T} \to \mathscr{R}$ be a morphism in **Prd**. For any $b \in \mathrm{ob}(H)$ and $h \in {}_bH$, one has $\nu_b(M_h) = N_{\alpha(h)}$. It follows

from the definitions that the homomorphism $\nu_b\colon \Gamma(b) \to \Lambda(\alpha(b))$ of Boolean rings restricts to a homomorphism $\nu'_b\colon \Gamma'(b) \to \Lambda'(\alpha(b))$. Clearly, the homomorphisms ν'_b for $b \in \mathrm{ob}(H)$ are the components of a natural transformation $\nu'\colon \Gamma' \to \Lambda'\alpha$, and $f' := (\alpha, \nu')\colon \mathscr{T}^a \to \mathscr{R}^a$ is a morphism in $\mathbf{Prd}$. Setting $\mathfrak{A}(f) := f'$ defines the functor $\mathfrak{A}$ as required.

It is easily seen that $\mathfrak{A}$ is right adjoint to the inclusion functor $\mathfrak{B}\colon \mathbf{Prd}^a \to \mathbf{Prd}$. The unit of the adjunction is the identity natural transformation of the identity functor $\mathrm{Id}_{\mathbf{Prd}^a}$. The component at $\mathscr{R} = (G, \Lambda, N)$ of the counit is the protorootoid morphism $(\mathrm{Id}_G, \mu)\colon \mathscr{R}^a \to \mathscr{R}$ in which for all $a \in \mathrm{ob}(G)$, μ_a is the inclusion $\Lambda'(a) \to \Lambda(a)$. In particular, $\mathbf{Prd}^a$ is a full, coreflective subcategory of $\mathbf{Prd}$.

3.11. The following result describes the relationship between principal and preprincipal protorootoids.

Proposition. *The protorootoid $\mathscr{R}$ is preprincipal if and only if its abridgement $\mathscr{R}^a$ is principal. In that case, the atomic generators of $\mathscr{R}$ coincide with the simple generators of $\mathscr{R}^a$.*

Proof. Let $\mathscr{R}$ be any protorootoid. Note that if $\mathscr{R}$ is preprincipal and $\mathscr{T} := \mathscr{R}^a$ is principal, then $S_{\mathscr{T}} = A_{\mathscr{T}} = A_{\mathscr{R}}$ by Lemma 3.7 and the comments at the end of 3.10. Those comments also show that (i)–(ii) below hold:

(i) If $\mathscr{R}$ is principal, then $\mathscr{R}^a$ is principal.
(ii) $\mathscr{R}$ is preprincipal if and only if $\mathscr{R}^a$ is preprincipal.
(iii) If $\mathscr{R}$ is principal, it is preprincipal.
(iv) If $\mathscr{R}$ is preprincipal and abridged, it is principal.

It is easily seen that (ii)–(iv) imply ((i) and) the first assertion of the Lemma, so we need only prove (iii)–(iv).

To prove (iii), assume that $\mathscr{R}$ is principal. Then $A := A_{\mathscr{R}} = S_{\mathscr{R}} =: S$ and $l_A = l_S = l_N$. For any $a \in \mathrm{ob}(G)$, $g \in {}_aG$ and $s \in {}_aA$, either $N(g) \cap N(s) = \emptyset$ or $N(s) \subseteq N(g)$ since $s \in S$ implies that $N(s)$ is an atom of ${}_a\Lambda$. Hence $\mathscr{R}$ is preprincipal, since it is faithful and interval finite.

To prove (iv), assume that $\mathscr{R} = (G, \Lambda, N)$ is preprincipal and abridged. Let $a \in \mathrm{ob}(G)$ and $s \in {}_aA$ where $A := A_{\mathscr{R}}$. Define

$$B := \{\, u \in {}_a\Lambda \mid u \cap N_s \in \{\emptyset, N_s\}\,\}. \tag{3.11.1}$$

Using the fact that $\{\emptyset, N_s\}$ is a subring of ${}_a\Lambda$, one easily checks that B is a subring of ${}_a\Lambda$. By the assumption that $\mathscr{R}$ is preprincipal, B contains ${}_a\mathscr{L}$. Since $\mathscr{R}$ is abridged, ${}_a\mathscr{L}$ generates ${}_a\Lambda$ as ring, so $B = {}_a\Lambda$. This implies that $N(s)$ is an atom of ${}_a\Lambda$ i.e. $s \in S := S_{\mathscr{R}}$. Thus $A \subseteq S$. Since $\mathscr{R}$ is preprincipal, it is faithful and interval finite, hence atomically generated by Lemma 3.5. By Lemma 3.7, it follows that $\mathscr{R}$ is principal. $\square$

3.12. The final characterization of principal protorootoids here is the following.

Lemma. *Let $\mathscr{R}$ be a protorootoid.*

(a) *$\mathscr{R}$ is principal if and only if it is preprincipal and saturated.*

(b) *$\mathscr{R}$ is preprincipal if and only if it is faithful, interval finite and pseudoprincipal.*

Proof. For the proof of (a), suppose first that $\mathscr{R} = (G, \Lambda, N)$ is principal. It is preprincipal by 3.11(iii). To show $\mathscr{R}$ is saturated, let $g \in {}_aG$. It will suffice to show that a maximal chain M in $[\emptyset, N(g)]_{{}_a\mathscr{L}}$ is also a maximal chain in $[\emptyset, N(g)]_{{}_a\Lambda}$. This is trivial if $g = 1_a$. Assume $g \neq 1_a$ and write M as

$$\emptyset \subsetneq N(x_1) \subsetneq N(x_1x_2) \subsetneq \ldots \subsetneq N(x_1 \cdots x_n), \quad g = x_1 \ldots x_n \tag{3.12.1}$$

in $[\emptyset, N(g)]_{{}_a\mathscr{L}}$. It is easy to show that each $x_i \in S := S_{\mathscr{R}}$ (for instance, from the correspondence of chains such as M with compatible expressions with value g, and the substitution property; see 2.6). Now any chain in the Boolean interval $[\emptyset, N(g)]_{{}_a\Lambda}$ has length at most $\mathrm{rank}(N(g)) = l_N(g)$, and any expression of g as a product of elements of S has length at least $l_S(g)$, so $l_S(g) \leq n \leq l_N(g)$. Since $\mathscr{R}$ is principal, $l_N(g) = l_S(g) = n$. This implies that the above chain is a maximal chain in $[\emptyset, N(g)]_{{}_a\Lambda}$, so $\mathscr{R}$ is saturated.

Conversely, suppose that $\mathscr{R}$ is preprincipal and saturated. Let $S := S_{\mathscr{R}}$, $A := A_{\mathscr{R}}$. By Lemma 3.5, Lemma 3.7 and the definition of preprincipal protorootoids, it will suffice to show that $A \subseteq S$. Let $a \in \mathrm{ob}(G)$ and $s \in {}_aS$. Then since $N(s)$ is an atom of ${}_a\mathscr{L}$, it follows that $\emptyset \subseteq N(s)$ is a maximal chain in $[\emptyset, N(s)]_{{}_a\mathscr{L}}$. Since $\mathscr{R}$ is saturated, this is also a maximal chain in $[\emptyset, N(s)]_{{}_a\Lambda}$ i.e. $N(s)$ is an atom of ${}_a\Lambda$. This shows $s \in S$, so $A \subseteq S$ as required to complete the proof of (a).

To prove (b), set $A := A_{\mathscr{R}}$. Suppose first that $\mathscr{R}$ is preprincipal. Then it is interval finite and faithful by assumption. Let $g, h \in {}_aG$ with $N(h) \neq \emptyset$. Since $\mathscr{R}$ is interval finite, there exists some $x \in {}_aA$, such that $\emptyset \subseteq N(x) \subseteq N(h)$. By definition of preprincipal rootoid, either $N(x) \cap N(g) = \emptyset$ or $N(x) \subseteq N(g)$, which implies that $\mathscr{R}$ is pseudoprincipal. Conversely, suppose that $\mathscr{R}$ is faithful, interval finite and pseudoprincipal. Let $g \in {}_aG$ and $s \in {}_aS$. Taking $h = s$ in the defining condition of pseudoprincipal protorootoid, there exists $x \in {}_aG$ with $\emptyset \neq N(x) \subseteq N(s)$, and either $N(x) \cap N(g) = \emptyset$, or $N(x) \subseteq N(g)$. Since s is an atom and $\mathscr{R}$ is faithful, it follows that $x = s$, and so $\mathscr{R}$ is preprincipal. $\square$

Remarks. The previous results show that principal protorootoids are interval finite, regular, saturated and pseudoprincipal. Regular, saturated, pseudoprincipal rootoids will be studied in subsequent papers as a generalization of principal rootoids.

3.13. In the case of cocycle finite, principal and preprincipal protorootoids, compatibility has the following descriptions in terms of length functions on G.

Lemma. *Let $\mathscr{R} = (G, \Lambda, N)$ be a protorootoid and $e = {}_{a_0}[g_1, \ldots, g_n]_{a_n}$ be an expression in G with value g.*

(a) *If $\mathscr{R}$ is cocycle finite, then e is compatible if and only if $l_N(g) = \sum_{i=1}^n l_N(g_i)$.*
(b) *If $\mathscr{R}$ is principal and $S := S_{\mathscr{R}}$, then e is compatible if and only if $l_S(g) = \sum_{i=1}^n l_S(g_i)$.*
(c) *If $\mathscr{R}$ is preprincipal and $A := A_{\mathscr{R}}$, then e is compatible if and only if $l_A(g) = \sum_{i=1}^n l_A(g_i)$.*

Proof. Part (a) is trivial for $n \le 1$. Suppose now that $n = 2$. Then by Lemma 3.2,

$$\begin{aligned} l_N(g_1g_2) &= \mathrm{rank}(N(g_1g_2)) = \mathrm{rank}(N(g_1) + g_1N(g_2)) \\ &= \mathrm{rank}(N(g_1) + \mathrm{rank}(g_1N(g_2)) - \mathrm{rank}(N(g_1) \cap g_1N(g_2)) \\ &= l_N(g_1) + l_N(g_2) - \mathrm{rank}(N(g_1) \cap g_1N(g_2)) \end{aligned}$$

since $[\emptyset, g_1N(g_2)] \cong [\emptyset, N(g_2)]$ implies that $\mathrm{rank}(g_1N(g_2)) = l_N(g_2)$. Thus $l_N(g_1g_2) = l_N(g_1) + l_N(g_2)$ if and only if $N(g_1) \cap g_1N(g_2) = \emptyset$, which holds if and only if g_1g_2 is compatible. In general, (a) follows from the $n = 2$ case by induction using the substitution property (Lemma 2.6). Part (b) follows from (a) since if $\mathscr{R}$ is principal, then $l_S = l_N$ by definition. Part (c) follows from (b) using Proposition 3.11. □

3.14. The following shows that, for principal or cocycle finite protorootoids, the weak preorder can be equivalently expressed in terms of the appropriate length functions in a manner similar to the standard definition for weak order on Coxeter groups (cf. [1]). Similarly, orthogonality of morphisms can be expressed in terms of length functions (see 2.7).

Corollary. *Let $\mathscr{R} := (G, \Lambda, N)$ be a protorootoid, $a \in \mathrm{ob}(G)$, $x, y \in {}_aG$ and set $z := x^*y$.*

(a) *If $\mathscr{R}$ is cocycle finite, then $x \;{}_a\!\le y$ if and only if $l_N(y) = l_N(x) + l_N(z)$.*
(b) *If $\mathscr{R}$ is principal and $S := S_{\mathscr{R}}$ is its set of simple generators, then $x \;{}_a\!\le y$ if and only if $l_S(y) = l_S(x) + l_S(z)$.*
(c) *If $\mathscr{R}$ is preprincipal and $A := A_{\mathscr{R}}$ is its set of simple generators, then $x \;{}_a\!\le y$ if and only if $l_A(y) = l_A(x) + l_A(z)$.*

Proof. This follows from Lemmas 2.7 and 3.13. □

4. Rootoids

4.1. **Complemented protomeshes.** A protomesh (R, L) is said to be *complemented* if R is unital and for each $A \in L$, $A^{\complement} := 1_R + A \in L$.

Lemma. *Let (R, L) be a complemented protomesh.*

(a) *Suppose given a family $(A_i)_{i \in I}$ of elements of L and $B \in L$ satisfying $A_i \cap B = \emptyset$ for all $i \in I$. If the join $A = \bigvee_i A_i$ exists in L, then $A \cap B = \emptyset$.*
(b) *For all $A \in R$, $(R, A + L)$ is a complemented protomesh.*
(c) *If $A \in L$, then $A + L$ has a maximum element 1_R.*

Proof. One has $A_i \subseteq B^{\complement} \in L$ for all i, so $A = \bigvee A_i \subseteq B^{\complement}$ and $A \cap B = \emptyset$. This proves (a). For (b), a typical element of $A + L$ is $A + B$ where $B \in L$. One has $B^{\complement} = 1_R + B \in L$, so $(A + B)^{\complement} = (A + B) + 1_R = A + (B + 1_R) \in A + L$. For (c), $1_R = A + A^{\complement} \in A + L$. □

4.2. **The JOP.** The following condition $(*)$ on a protomesh (R, L), which is suggested by Lemma 4.1(a), is called the *join orthogonality property (JOP).*

$(*)$ Suppose that $(A_i)_{i \in I}$ is a family of elements of L and $B \in L$ satisfies $A_i \cap B = \emptyset$ for all $i \in I$. If the join $A = \bigvee_i A_i$ exists in L, then $A \cap B = \emptyset$.

By abuse of terminology, we may say that L satisfies the JOP when $(*)$ holds.

4.3. **Rootoids.** Rootoids are defined as faithful protorootoids such that their weak orders are complete meet semilattices which satisfy the JOP. In full detail:

Definition. A protorootoid $\mathscr{R} := (G, \Lambda, N)$ with big weak order $\mathscr{L}$ is said to be a *rootoid*, or to be *rootoidal*, if it satisfies the following conditions (i)–(iii):

(i) For all $a \in \mathrm{ob}(G)$, if $g, h \in {}_aG$ with $N(g) = N(h)$, then $g = h$.
(ii) For all $a \in \mathrm{ob}\, G$, ${}_a\mathscr{L} := \{ N(g) \mid g \in {}_aG \}$ is a complete meet semilattice.
(iii) If $a \in \mathrm{ob}\, G$, and $A_i, A \in {}_a\mathscr{L}$ are such that $A_i \cap A = \emptyset$ for all $i \in I$ and $B := \bigvee_i A_i$ exists in ${}_a\mathscr{L}$, then $B \cap A = \emptyset$.

A protorootoid satisfying (iii) is said to satisfy JOP, even if it does not satisfy (i)–(ii). Rootoids are said to have a certain property of protorootoids if the underlying protorootoid has that property. For example, a rootoid is *complete* (resp., *principal*) if it is complete (resp., principal) as a protorootoid. Thus, a rootoid is complete if and only if each of its weak orders has a maximum element. If $\mathscr{R}$ is a complemented rootoid, then each weak order ${}_a\mathscr{L}$ is a complete ortholattice and in particular, $\mathscr{R}$ is complete.

Remarks. (1) The conditions for a protorootoid to be a rootoid depend only on its preorder isomorphism type, since (iii) may be reformulated in terms of orthogonality of morphisms in G.

(2) In particular, (1) implies that a protorootoid is a rootoid if and only if its abridgement is a rootoid, since abridgement doesn't change the underlying groupoid-preorder.

4.4. Let $\mathscr{R} = (G, \Lambda, N)$ denote a protorootoid with big weak order $\mathscr{L}$. Part (c) of the following is a technical consequence of the JOP which plays an important role in subsequent papers.

Lemma. *Let $x \in {}_aG_b$.*

(a) *The map $A \mapsto B := x \cdot A = N_x \,\dot\cup\, x(A)$ induces an order isomorphism $\{ A \in {}_b\mathscr{L} \mid A \cap N_{x^*} = \emptyset \} \xrightarrow[\theta]{\cong} \{ B \in {}_a\mathscr{L} \mid B \supseteq N_x \}$.*
(b) *The maps θ and θ^{-1} preserve whatever meets and joins exist in their domains (in the natural order induced by the appropriate weak order).*
(c) *If $\mathscr{R}$ is a rootoid, then $\mathrm{dom}(\theta)$ (resp., $\mathrm{cod}(\theta)$) is a complete join-closed meet subsemilattice of ${}_b\mathscr{L}$ (resp., ${}_a\mathscr{L}$). Hence θ (resp., θ^{-1}) preserves meets or joins (of non-empty subsets of its domain) which exist in ${}_b\mathscr{L}$ (resp., in ${}_a\mathscr{L}$).*

Remarks. The point of (c) in relation to (b) is that joins or meets of non-empty subsets of $\mathrm{dom}(\theta)$ (resp., $\mathrm{cod}(\theta)$) exist or not, and have the same value if they exist, whether taken in $\mathrm{dom}(\theta)$ or in ${}_b\mathscr{L}$ (resp., in $\mathrm{cod}(\theta)$ or in ${}_a\mathscr{L}$).

Proof. Suppose $A \in {}_b\mathscr{L}$ with $A \cap N_{x^*} = \emptyset$. Let $B := x \cdot A = N_x + x(A)$. Since $N(x) \cap x(A) = x(N_{x^*} \cap A) = \emptyset$, $B = N_x \,\dot\cup\, x(A)$ is in the right hand side and $\theta\colon A \mapsto x \cdot A = B$ is order-preserving. Similarly, if $B \in {}_a\mathscr{L}$ with $B \supseteq N_x$, then $A := x^* \cdot B = N_{x^*} + x^*(B)$ where $N_{x^*} = x^*(N_x) \subseteq x^*(B)$, so A is in the left hand side and $\theta^{-1}\colon B \mapsto x^* \cdot B = A$ is order preserving. This proves (a). The assertions of (b) are true of any order isomorphism whatsoever.

For (c), $\mathrm{dom}(\theta)$ is closed under taking meets of its non-empty subsets in ${}_b\mathscr{L}$ since it is an order ideal of ${}_b\mathscr{L}$. Also, $\mathrm{dom}(\theta)$ is closed under taking those joins which exist in ${}_b\mathscr{L}$ of its non-empty subsets since $\mathscr{R}$ satisfies JOP. Similarly, $\mathrm{cod}(\theta)$ is closed under taking meets in ${}_a\mathscr{L}$ of non-empty subsets of $\mathrm{cod}(\theta)$, since $N_x \in {}_a\mathscr{L}$ is a lower bound for any subset of $\mathrm{cod}(\theta)$. Finally, $\mathrm{cod}(\theta)$ is closed under joins which exist in ${}_a\mathscr{L}$ of non-empty subsets of $\mathrm{cod}(\theta)$ since $\mathrm{cod}(\theta)$ is an order coideal of ${}_a\mathscr{L}$. □

4.5. A semilattice X is said to be *pseudocomplemented* if it has a minimum element 0_X and for any $x \in X$, there exists $x' \in X$ such that for $y \in X$, $y \bigwedge x = 0_X$ if and only if $y \leq x'$.

Proposition. *Let $\mathscr{R} = (G, \Lambda, N)$ be a principal rootoid with simple generators $S := S_{\mathscr{R}}$. Let $a \in \mathrm{ob}(G)$.*

(a) *For all $x \in {}_aG$, let $S(x) := \{\, s \in {}_aS \mid N_s \subseteq N_x \,\}$. Then for a non-empty family (x_i) in ${}_aG$, one has $\bigwedge x_i = 1_a$ if and only if $\bigcap_i S(x_i) = \emptyset$.*
(b) *If $x_i, y \in {}_aG$ are such that $x_i \wedge y = 1_a$ for all $i \in I \neq \emptyset$ and $x := \bigvee_i x_i$ exists in ${}_aG$, then $x \wedge y = 1_a$.*
(c) *If $\mathscr{R}$ is complete, then its weak orders are pseudocomplemented.*

Proof. Part (a) follows on noting that

$$S\big(\bigwedge x_i\big) = \bigcap S(x_i) \tag{4.5.1}$$

and that, for $x \in {}_aG$, $S(x) = \emptyset$ if and only if $x = 1_a$. For part (b), note first that for $s \in {}_aS$ and $x \in {}_aG$, one has either $N_s \subseteq N_x$ or $N_s \cap N_x = \emptyset$, since N_s is an atom of ${}_a\Lambda$. The JOP therefore implies that

$$S\big(\bigvee x_i\big) = \bigcup_i S(x_i). \tag{4.5.2}$$

Part (b), which is formally similar to JOP, follows readily from (4.5.1), (4.5.2) and (a). To prove (c), one checks from (b) that if $x \in {}_aG$, the pseudocomplement of x is

$$x' := \bigvee_{\substack{y \in {}_aG \\ y \wedge x = 1_a}} y. \tag{4.5.3}$$
□

4.6. **Complete semilattices.** Subsections 4.6–4.7 describe basic facts about a category of complete semilattices which is involved in the definition in 4.8 of the main categories of rootoids considered in these papers.

Recall that the category **PreOrd** denotes the category of preordered sets with morphisms given by preorder-preserving maps, and **Ord** is its full subcategory of posets. One may view **PreOrd** as a full subcategory of **Cat**, with objects those small categories for which there is at most one morphism between any two objects. Then **Ord** is the full subcategory of **PreOrd** with objects the preordered sets (as categories) in which every isomorphism is an identity map.

The category $\mathbf{CSL}_0$ is the following subcategory of **Ord**. The objects Γ of $\mathbf{CSL}_0$ are the non-empty complete meet semilattices Γ, viewed as non-empty posets in which every non-empty subset has a greatest lower bound. A morphism $\theta \colon \Lambda \to \Gamma$ in $\mathbf{CSL}_0$ is a function $\Lambda \to \Gamma$, preserving minimum elements, which also preserves

all meets of non-empty subsets of Λ and all joins which exist of subsets of Λ. That is, $\theta(0_\Lambda) = 0_\Gamma$, $\theta(\bigwedge X) = \bigwedge \theta(X)$ for all $\emptyset \neq X \subseteq \Lambda$, $\theta(\bigvee Y) = \bigvee(\theta Y)$ for any $Y \subseteq \Lambda$ for which $\bigvee Y$ exists. Such a morphism is order-preserving i.e. has an underlying morphism in **Ord** (to see this, note that $x \leq y$ if and only if $x \bigwedge y = x$, in any poset).

Remarks. If $\theta\colon \Lambda \to \Gamma$ is a morphism in $\mathbf{CSL}_0$ and the underlying map of sets $\Lambda \to \Gamma$ is bijective, then θ is an isomorphism in $\mathbf{CSL}_0$. This follows by first using the preceding criterion for $x \leq y$ to show that θ is an isomorphism in **Ord**.

4.7. Let $\theta\colon \Lambda \to \Gamma$ be a morphism in $\mathbf{CSL}_0$. Let

$$\Gamma' := \{\, \gamma \in \Gamma \mid \gamma \leq \theta(\alpha) \text{ for some } \alpha \in \Lambda \,\} \tag{4.7.1}$$

denote the order ideal of Γ generated by the image of θ. Then Γ' with the induced order is an object of $\mathbf{CSL}_0$ and θ restricts to a morphism $\theta'\colon \Lambda \to \Gamma'$ in $\mathbf{CSL}_0$. Note that, viewing Λ, Γ' as categories and θ' as a functor, θ' has a left adjoint $\theta^\perp\colon \Gamma' \to \Lambda$. That is, $\theta^\perp$ is a functor (order preserving map) $\Gamma' \to \Lambda$ satisfying

$$\mathrm{Hom}_{\Gamma'}(\gamma, \theta'(\alpha)) \cong \mathrm{Hom}_\Lambda(\theta^\perp(\gamma), \alpha) \tag{4.7.2}$$

for all $\alpha \in \Lambda$ and $\gamma \in \Gamma'$. In fact, $\theta^\perp$ is uniquely determined by the corresponding map of objects, which is given by

$$\theta^\perp(\gamma) := \bigwedge\{\, \alpha \in \Lambda \mid \gamma \leq \theta(\alpha) \,\} = \min(\{\, \alpha \in \Lambda \mid \gamma \leq \theta(\alpha) \,\}) \tag{4.7.3}$$

for $\gamma \in \Gamma'$. One has

$$\gamma \leq \theta(\alpha) \iff \theta^\perp(\gamma) \leq \alpha \tag{4.7.4}$$

for all $\gamma \in \Gamma'$ and $\alpha \in \Lambda$. As a left adjoint, $\theta^\perp$ preserves colimits, such as coproducts. Hence its underlying map of objects $\Gamma' \to \Lambda$ preserves those joins which exist in Γ', and is order-preserving.

Henceforward, for any morphism θ in $\mathbf{CSL}_0$, $\theta^\perp$ will be identified with its corresponding map of objects, unless otherwise specified i.e. it will regarded as a function $\theta^\perp\colon \Gamma' \to \Lambda$ where $\Gamma' := \mathrm{dom}(\theta^\perp)$. Informally, it is often convenient to regard $\theta^\perp$ as a partially defined, join preserving map $\Gamma \to \Lambda$, defined only at elements of the order ideal of Γ generated by the image of θ. The following Lemma, which is a trivial variant of well known facts about adjoints of composite functors, implies that $\theta \mapsto \theta^\perp$ gives a contravariant functor to a suitable category of complete meet semilattices with partially defined, join preserving maps.

Lemma. *Let $\theta_i\colon \Lambda_{i-1} \to \Lambda_i$ be morphisms in $\mathbf{CSL}_0$ for $i = 1, 2$. Let $\theta := \theta_2\theta_1\colon \Lambda_0 \to \Lambda_2$. Then $\mathrm{dom}(\theta^\perp) = \{\, \gamma \in \mathrm{dom}(\theta_2^\perp) \mid \theta_2^\perp(\gamma) \in \mathrm{dom}(\theta_1^\perp) \,\}$ and for $\gamma \in \mathrm{dom}(\theta^\perp)$, one has $(\theta_2\theta_1)^\perp(\gamma) = \theta_1^\perp(\theta_2^\perp(\gamma))$.*

Proof. For $\gamma \in \Lambda_2$ and $\alpha \in \Lambda_0$, one has

$$(\theta_2\theta_1)^\perp(\gamma) \leq \alpha \iff \gamma \leq \theta_2(\theta_1(\alpha)) \iff \theta_2^\perp(\gamma) \leq \theta_1(\alpha) \iff \theta_1^\perp(\theta_2^\perp(\gamma)) \leq \alpha$$

assuming all terms involved are defined. The remaining details are omitted. □

4.8. **Categories of rootoids.** The categories **Rd** and **RdE** of rootoids defined below are those of primary concern. They are defined as subcategories of an auxiliary category **rd**. The category **Rd** (resp, **RdE**) is called the category of *rootoids* (resp., of *rootoid local embeddings*). The term "morphism of rootoids" (resp., "local embedding of rootoids") refer to a morphism of **Rd** (resp., **RdE**) unless otherwise specified.

Definition. (a) The category **rd** is the following subcategory of **Prd**. Its objects are rootoids (G, L, N). For any object (G, L, N) of **rd** and each $a \in \mathrm{ob}(G)$, regard $({}_aG, {}_a\leq)$ as an object of $\mathbf{CSL}_0$. Morphisms in **rd** are morphisms $(\alpha, \mu)\colon (G, \Lambda, N) \to (G', \Lambda', N')$ in **Prd** such that for each $a \in \mathrm{ob}(G)$, the map ${}_a\alpha\colon {}_aG \to {}_{\alpha(a)}G'$ induced by α is a morphism in $\mathbf{CSL}_0$.

(b) The category **Rd** of rootoids is the following subcategory of **rd**. It has the same objects as **rd**, and a morphism $(\alpha, \mu)\colon (G, \Lambda, N) \to (G', \Lambda', N')$ in **rd** is a morphism in **Rd** if and only if for all $a \in \mathrm{ob}(G)$, $g \in {}_aG$ and $g' \in {}_{\alpha(a)}G'$ with g' and ${}_a\alpha(g)$ orthogonal in G' (i.e. $N'({}_a\alpha(g)) \cap N'_{g'} = \emptyset$ in ${}_{\alpha(a)}\Lambda'$) and with $g' \in \mathrm{dom}({}_a\alpha^{\perp})$, one has g and ${}_a\alpha^{\perp}(g')$ orthogonal in G (i.e. $N(g) \cap N({}_a\alpha^{\perp}(g')) = \emptyset$ in ${}_a\Lambda$).

(c) The category **RdE** of local embeddings of rootoids is the subcategory of **Rd** with all objects, and morphisms $(\alpha, \mu)\colon (G, \Lambda, N) \to (G', \Lambda', N')$ in **Rd** such that for each $a \in \mathrm{ob}(G)$, the map ${}_a\alpha\colon {}_aG \to {}_{\alpha(a)}G'$ in $\mathbf{CSL}_0$ induced by α is injective and its image is a join-closed meet subsemilattice of ${}_{\alpha(a)}G'$.

It is straightforward to check that **rd**, **Rd** and **RdE** are categories; the fact that composites of morphisms in **Rd** are again morphisms in **Rd** is a consequence of Lemma 4.7 and the definitions.

The additional condition required in (b) for a morphism $f = (\alpha, \mu)$ of **rd** to be a morphism in **Rd** will be called the *adjunction orthogonality property* or *AOP* for short. Thus, a morphism in **Rd** is a morphism in **rd** satisfying AOP. Notice that whether a morphism $f = (\alpha, \mu)$ in **Prd** is a morphism in **rd**, **Rd** or **RdE** depends only on the underlying morphism $\alpha = \mathfrak{P}(f)$ of groupoid-preorders.

4.9. **Coverings.** The following Lemma lists useful properties of covering morphisms.

Lemma. *Let $f = (\alpha, \nu)\colon \mathscr{R}' \to \mathscr{R}$ be a covering morphism of protorootoids. Write $\mathscr{R} = (G, \Lambda, N)$ and $\mathscr{R}' = (H, \Lambda', N')$. Let $A := A_{\mathscr{R}}$, $A' := A_{\mathscr{R}'}$, $S := A_{\mathscr{R}}$, $S' := A_{\mathscr{R}'}$ denote the sets of atomic and simple morphisms of $\mathscr{R}$ and $\mathscr{R}'$.*

(a) *The weak order of $\mathscr{R}'$ at an object a of H identifies naturally with the weak order of $\mathscr{R}$ at the object $\alpha(a)$ of G.*

(b) *For $a \in \mathrm{ob}(G)$, one has ${}_aA' = \{\, s \in {}_aH \mid \alpha(s) \in {}_{\alpha(a)}A \,\}$ and ${}_aS' = \{\, s \in {}_aH \mid \alpha(s) \in {}_{\alpha(a)}S \,\}$. If A (resp., S) generates G, then A' (resp., S') generates H; the converses hold if f is a covering quotient morphism.*

(c) *If $\mathscr{R}$ is a (faithful, complete, interval finite, cocycle finite, preprincipal, principal, pseudoprincipal, regular or rootoidal) protorootoid, then so is $\mathscr{R}'$; the converses hold if f is a covering quotient morphism.*

(d) *If $\mathscr{R}$ is a rootoid then f is a morphism in* **Rd** *and in* **RdE**.

Remarks. Note that it follows from (d) that an isomorphism in **Prd** between two rootoids is an isomorphism in **Rd** and **RdE**.

Proof. Let $\mathscr{L}'$ denote the big weak order of $\mathscr{R}'$. The map ν_a gives an isomorphism ${}_a\Lambda' \xrightarrow{\cong} {}_{\alpha(a)}\Lambda$ of Boolean rings such that for $h \in {}_aH$, $\nu_a(N'(h)) = N(\alpha(h))$. Since the map $h \mapsto \alpha(h) \colon {}_aH \to {}_{\alpha(a)}G$ is bijective by definition of covering morphism, the definitions imply, slightly more strongly than (a), that ν_a induces an isomorphism of protomeshes $({}_a\Lambda', {}_a\mathscr{L}') \cong ({}_{\alpha(a)}\Lambda, {}_{\alpha(a)}\mathscr{L})$.

We prove the part of (b) concerning simple morphisms. Note first that (a) and its proof imply that

$$l_{N'}(h) = l_N(\alpha(h)) \text{ for all } h \in \operatorname{mor}(H). \tag{4.9.1}$$

Hence

$$S' = \{\, s' \in \operatorname{mor}(H) \mid \alpha(s') \in S \,\}. \tag{4.9.2}$$

Using the definition of covering morphism, it follows that if S generates G, then S' generates H, and conversely if $\operatorname{ob}(H) \to \operatorname{ob}(G)$ is surjective. Assume that S generates G. A simple argument using (4.9.2) and the definition of covering morphism shows that

$$l_{S'}(h) = l_S(\alpha(h)) \text{ for all } h \in \operatorname{mor}(H). \tag{4.9.3}$$

The desired conclusions in (b) involving simple morphisms follows readily from these facts and the definitions. The proof of the parts of (b) involving atomic morphisms are similar using

$$A' = \{\, s' \in \operatorname{mor}(H) \mid \alpha(s') \in A \,\} \tag{4.9.4}$$

and, when A generates G,

$$l_{A'}(h) = l_A(\alpha(h)) \text{ for all } h \in \operatorname{mor}(H). \tag{4.9.5}$$

Part (c) follows easily from (a)–(b) and the above formulae, by the definitions.

Now for (d). There are inclusion functors

$$\mathbf{RdE} \longrightarrow \mathbf{Rd} \longrightarrow \mathbf{rd} \longrightarrow \mathbf{FPrd} \longrightarrow \mathbf{Prd}. \tag{4.9.6}$$

It will be shown that f is a morphism in each of these categories, working from right to left. By assumption, f is a morphism is **Prd** Next, f is an morphism is **FPrd**, since $\mathscr{R}'$ is faithful by (c) and **FPrd** is a full subcategory of **Prd**. Also by (c), $\mathscr{R}'$ is a rootoid since $\mathscr{R}$ is a rootoid. For any $a \in \operatorname{ob}(H)$, the map ${}_a\alpha \colon {}_aH \to {}_{\alpha(a)}G$ is an order isomorphism by (a). Since ${}_{\alpha(a)}G$ is in $\mathbf{CSL}_0$, it follows that ${}_a\alpha$ is an isomorphism in $\mathbf{CSL}_0$, and hence f is a morphism in **rd**; further, if f is a morphism in **Rd**, it is a morphism in **RdE**. To see that f is in **Rd**, we must verify AOP. Let $a, b \in \operatorname{ob}(H)$, $h \in {}_aH_b$ and $g \in {}_{\alpha(a)}G$ with g and ${}_a\alpha(h)$ orthogonal in G i.e. $(\alpha_a(h))^{-1} \;{}_{\alpha(b)}\!\leq (\alpha_a(h))^{-1}g$. Now ${}_b\alpha$ and ${}_a\alpha$ are order isomorphisms. Applying ${}_b\alpha^{-1}$ to the last equation gives $h^{-1} \;{}_b\!\leq h^{-1}g'$ where $g' = ({}_a\alpha)^{-1}(g) = {}_a\alpha^{\perp}(g)$. Hence h and g' are orthogonal in G as required. This completes the proof of (d). □

5. Root systems and set protorootoids

5.1. **Signed sets.** The *sign group* $\{\pm\}$ is the group of order two with elements $\{+,-\}$ and identity element $+$. Sometimes it is identified with the group $\{\pm 1\}$ of units of the ring $\mathbb{Z}$ of integers.

An action of a group H on a set Θ is said to be *free* if the stabilizers in H of all elements of Θ are trivial. An *indefinitely signed set* is defined to be a set Θ together with a free left action of the sign group. A *definitely signed set* is a pair (Θ,Θ_+) of an indefinitely signed set Θ together with a specified set $\Theta_+\subseteq\Theta$ of $\{\pm\}$-orbit representatives on Θ. Then $\Theta=\Theta_+\dot\cup\Theta_-$ where $\Theta_-:=-\Theta_+=\{\,-s\mid s\in\Theta_+\,\}$.

Let $\mathbf{Set}$ (resp., $\mathbf{Set}_{\{\pm\}}$, $\mathbf{Set}_{\pm}$) denote the categories of sets (resp., indefinitely signed sets, resp., definitely signed sets) with functions (resp., $\{\pm\}$-equivariant functions, $\{\pm\}$-equivariant functions) as morphisms.

Note especially that it is not required that a morphism $\Theta\to\Theta'$ in $\mathbf{Set}_{\pm}$ be *positivity preserving* i.e. that it induce a map $\Theta_+\to\Theta'_+$ (and therefore map Θ_- to Θ'_-). The subcategory of $\mathbf{Set}_{\pm}$ consisting of all its objects but with only positivity preserving morphisms will be denoted $\mathbf{Set}_{+,-}$. There is an equivalence of categories $\mathbf{Set}_{+,-}\xrightarrow{\cong}\mathbf{Set}$ given on objects by $\Theta\mapsto\Theta_+$.

5.2. **Signed groupoid-sets.** The category $\mathbf{Gpd\text{-}Set}_{\pm}$ has as its objects the pairs $R=(G,\Phi)$ where G is a groupoid and $\Phi\colon G\to\mathbf{Set}_{\pm}$ is a functor. Its objects are called *signed groupoid-sets.* A morphism $(G,\Phi)\to(H,\Psi)$ is defined to be a pair (α,ν) where $\alpha\colon G\to H$ is a functor and $\nu\colon\Psi\alpha\to\Phi$ is a natural transformation of functors $G\to\mathbf{Set}_{\pm}$ such that for each $a\in\mathrm{ob}(G)$, the component $\nu_a\colon\Psi(\alpha(a))\to\Phi(a)$ is *positivity preserving* i.e. each component is a morphism in the subcategory $\mathbf{Set}_{+,-}$ of $\mathbf{Set}_{\pm}$. For another morphism $(\beta,\mu)\colon(H,\Psi)\to(K,\Lambda)$, the composite $(\beta,\mu)(\alpha,\nu)\colon(G,\Phi)\to(K,\Lambda)$ is defined by $(\beta,\mu)(\alpha,\nu)=(\beta\alpha,\nu(\mu\alpha))$ where the component of $\nu(\mu\alpha)$ at $a\in\mathrm{ob}(G)$ is $(\nu(\mu\alpha))_a=\nu_a\mu_{\alpha(a)}$.

For fixed G, the subcategory G-$\mathbf{Set}_{\pm}$ of $\mathbf{Gpd\text{-}Set}_{\pm}$ containing objects (G,Φ) and morphisms (Id_G,ν) is called the category of signed G-sets.

The functor Φ is called the *root system* of a signed groupoid-set (G,Φ). It may sometimes be regarded as a family of definitely signed sets $({}_a\Phi)_{a\in\mathrm{ob}(G)}$ (or, if G is a group, as the unique signed set in that family) with action maps ${}_aG_b\times{}_b\Phi\to{}_a\Phi$ satisfying suitable conditions (associativity, inverse and unit axioms) similar to those for H-sets for a group H.

An element of ${}_a\Phi$ (resp., ${}_a\Phi_+$, ${}_a\Phi_-$) for some $a\in\mathrm{ob}(G)$, is called a *root* (resp., *positive root*, *negative root*) of R or Φ.

Remarks. (1) Note that morphisms of signed sets induced by the action of groupoid elements in a signed groupoid-set are *not* positivity preserving in general.

(2) There is also a (different) category $\mathbf{Gpd\text{-}Set}'_{\pm}$ with signed groupoid-sets as objects in which a morphism $(G,\Phi)\to(H,\Psi)$ is a pair (α,ν) where $\alpha\colon G\to H$ is a functor and $\nu\colon\Phi\to\Psi\alpha$ is a natural transformation with positivity preserving components. Morphisms in this latter category induce commutative diagrams of

action maps

$$
\begin{tikzcd}
{}_aG_b \times {}_b\Phi \arrow[r] \arrow[d] & {}_a\Phi \arrow[d] \\
{}_{\alpha(a)}H_{\alpha(b)} \times {}_{\alpha(b)}\Psi \arrow[r] & {}_{\alpha(a)}\Psi
\end{tikzcd}
\tag{5.2.1}
$$

However, it is **Gpd-Set**$_{\pm}$ in which the usual product (see [3])

$$(W_1, \Phi_1) \times (W_2, \Phi_2) = (W_1 \times W_2, \Phi_1 \coprod \Phi_2) \tag{5.2.2}$$

of root systems of Coxeter groups may be interpreted as a categorical product, and which is related to the category **Prd**; **Gpd-Set**$'_{\pm}$ is similarly related to **Prd**$'$.

5.3. The category of set protorootoids defined below provides a convenient bridge between the categories **Prd** of protorootoids and **Gpd-Set**$_{\pm}$ of signed groupoid-sets.

Definition. (a) A set protorootoid is defined to be a triple (G, Λ, N) such that G is a groupoid, $\Lambda\colon G \to \mathbf{Set}$ is a representation of G in **Set** and N is a G-cocycle for $\wp_G(\Lambda)$.

(b) A set protorootoid (G, Λ, N) is called a *set rootoid* if $(G, \wp_G(\Lambda), N)$ is a rootoid.

(c) The category **Set-Prd** has set protorootoids (G, Λ, N) as objects. A morphism $(G, \Lambda, N) \to (H, \Gamma, M)$ in **Set-Prd** is a pair (α, ν) consisting of a groupoid homomorphism $\alpha\colon G \to H$ and a natural transformation $\nu\colon \Gamma\alpha \to \Lambda$ such that for any $g \in {}_aG$, $M_{\alpha(g)} = \nu_a^{-1}(N_g) := \{\, p \in {}_{\alpha(a)}\Gamma \mid \nu_a(p) \in N_g \,\}$. Composition of morphisms is given by $(\beta, \mu)(\alpha, \nu) = (\beta\alpha, \nu(\mu\alpha))$.

5.4. There is a functor $\mathfrak{I}\colon \mathbf{Set\text{-}Prd} \to \mathbf{Prd}$ as follows. Directly from the definitions, if $\mathscr{R} = (G, \Lambda, n)$ is a set protorootoid, then $\mathfrak{I}(\mathscr{R}) := (G, \wp_G(\Lambda), N)$ is a protorootoid. Further, if $(\alpha, \nu)\colon (G, \Lambda, N) \to (H, \Gamma, M)$ is a morphism of set protorootoids, then

$$(\alpha, \wp_G(\nu))\colon (G, \wp_G(\Lambda), N) \to (H, \wp_H(\Gamma), M)$$

is a morphism in **Prd**. To see this, note that in 5.3(c), $\nu_a^{-1}(N_g) = (\wp_G(\nu))_a(N_g)$ and that $\wp_G(\Gamma\alpha) = \wp\Gamma\alpha\iota_G = \wp\Gamma\iota_H\alpha = \wp_H(\Gamma)\alpha$. This defines $\mathfrak{I}$ on objects and morphisms. Using $\wp_G(\nu\alpha) = \wp_G(\nu)\alpha$, one checks $\mathfrak{I}$ is a functor as claimed. Note that $\mathfrak{I}$ is faithful since $\wp$ is faithful.

Proposition. *Let $\mathscr{R}$ be a principal protorootoid. Then there is a set protorootoid $\mathscr{T}$ such that $\mathfrak{A}(\mathfrak{I}(\mathscr{T})) \cong \mathfrak{A}(\mathscr{R})$.*

Proof. Write $\mathscr{R} = (G, \Lambda, N)$ and denote its simple generators as S and abridgement $\mathscr{R}^a$ as (G, Λ', N'). The cocycle condition implies that for $a \in \mathrm{ob}(G)$, $\Lambda'(a)$ is generated as Boolean ring by the elements $g(N_s)$ for $g \in {}_aG_b$ and $s \in {}_bS$. Let ${}_a\Phi$ denote the set of all these elements. Note that elements of ${}_a\Phi$ are atoms of $\Lambda(a)$, hence of $\Lambda'(a)$. Thus, ${}_a\Phi$ is a set of orthogonal idempotents generating ${}_a\Lambda'$, which is therefore isomorphic to the Boolean ring of finite subsets of ${}_a\Phi$. Note that there is a natural representation of G on the sets ${}_a\Phi$, corresponding

to a functor $\Phi\colon G\to\mathbf{Set}$ with $\Phi(a)={}_a\Phi$. Clearly, Λ' is equivalent to the subrepresentation $\wp'_G(\Phi)$ of $\wp_G(\Phi)$ such that ${}_a(\wp'_G(\Phi))$ is the set of all finite subsets of ${}_a(\wp_G(\Phi))$. Let $M\colon G\to\dot\cup_{a\in\mathrm{ob}(G)}{}_a(\wp_G(\Phi))$ be the function defined by $M(g):=\{\,x\in{}_a\Phi\mid x\subseteq N'(g)\,\}\in{}_a(\wp'_G(\Phi))$. Using Lemma 3.2, M is a G-cocycle for $\wp_G(\Phi)$. It is easy to see from this that $\mathscr{T}:=(G,\Phi,M)$ is a set protorootoid with the required property. $\square$

Remarks. One version of Stone's theorem implies that any Boolean ring has a (canonical) realization as a ring of sets. It is easy to see from it that any protorootoid is preorder isomorphic to a protorootoid in the image of $\mathfrak{I}$. A more precise statement of this fact, and a generalization of the above proposition, will be proved in a subsequent paper.

5.5. The following result is a special case (involving only fibers $\{\pm\}$) of facts about the analogues of bundles in the category of groupoid representations in the category of sets (cf. [8] for related results for groups).

Proposition. *The categories* **Set-Prd** *and* $\mathbf{Gpd\text{-}Set}_{\pm}$ *are equivalent.*

Proof. We first construct a functor $\mathfrak{K}\colon\mathbf{Set\text{-}Prd}\to\mathbf{Gpd\text{-}Set}_{\pm}$. Let (G,Λ,N) be a set protorootoid. Define a functor $\Phi\colon G\to\mathbf{Set}_{\pm}$ as follows. For $a\in\mathrm{ob}(G)$, let $\Phi(a)={}_a\Phi:={}_a\Lambda\times\{\pm\}$. Regard it as a definitely signed set with $\{\pm\}$ action, which we write as $(\epsilon,\alpha)\mapsto\epsilon\alpha$, by multiplication on the second factor and with ${}_a\Phi_\epsilon:={}_a\Lambda\times\{\epsilon\}$ for $\epsilon\in\{\pm\}$ and $a\in\mathrm{ob}(G)$. For $a\in\mathrm{ob}(G)$, $g\in{}_aG_b$, $x\in{}_b\Lambda$ and $\epsilon\in\{\pm\}$, set

$$\Phi(g)(x,\epsilon):=\big((\Lambda(g))(x),\epsilon\epsilon'\big),\quad \epsilon'=\epsilon'_{g,x}:=\begin{cases}-, & x\in N(g^*)\\ +, & x\notin N(g^*).\end{cases}\tag{5.5.1}$$

One can check this defines a signed groupoid-set (G,Φ), and we set $\mathfrak{K}(G,\Lambda,N)=(G,\Phi)$. Next, suppose given a morphism $(\alpha,\nu)\colon(G,\Lambda,N)\to(H,\Gamma,M)$. Write $\mathfrak{K}(H,\Gamma,M)=(H,\Psi)$. Define a natural transformation $\nu'\colon\Psi\alpha\to\Phi$ which has component $\nu'_a\colon\Psi\alpha(a)\to\Phi(a)$ at a equal to the map

$$\nu_a\times\mathrm{Id}_{\{\pm\}}\colon{}_{\alpha(a)}\Gamma\times\{\pm\}\to{}_a\Lambda\times\{\pm\}.$$

Using (5.5.1), it can be checked that $(\alpha,\nu')\colon(G,\Phi)\to(H,\Psi)$ is a morphism in $\mathbf{Gpd\text{-}Set}_{\pm}$ and that setting $\mathfrak{K}(\alpha,\nu)=(\alpha,\nu')$ defines a functor $\mathfrak{K}$ as required.

A functor $\mathfrak{L}\colon\mathbf{Gpd\text{-}Set}_{\pm}\to\mathbf{Set\text{-}Prd}$ defining an inverse equivalence may be constructed as follows. Suppose that (G,Φ) is a signed groupoid-set. For $a\in\mathrm{ob}(G)$, let ${}_a\Lambda:=({}_a\Phi)/\{\pm1\}$ be the $\{\pm\}$-orbit space. Since morphisms of signed G-sets are $\{\pm\}$-equivariant, there is a natural functor $\Lambda\colon G\to\mathbf{Set}$ induced by Φ with $\Lambda(a)={}_a\Lambda$ for all $a\in\mathrm{ob}(G)$. Let $\pi_a\colon{}_a\Phi\to{}_a\Lambda$ be the orbit map $\alpha\mapsto\{\pm\alpha\}$. For $g\in{}_aG_b$, define

$${}_aN'(g):={}_a\Phi_++\Phi(g)({}_b\Phi_+)\in\wp({}_a\Phi).\tag{5.5.2}$$

This defines a G-cocycle (in fact, a coboundary) N' for $\wp_G(\Phi)$. Note that

$${}_aN'(g)=\big({}_a\Phi_+\cap\Phi(g)({}_b\Phi_-)\big)\,\dot\cup\,\big({}_a\Phi_-\cap\Phi(g)\big({}_b\Phi_+\big)\big)\tag{5.5.3}$$

and in particular $N'(g) = \{ -\alpha \mid \alpha \in N'(g) \}$. Let

$$ {}_aN(g) := \pi_a(N'_g) = \pi_a\big({}_a\Phi_+ \cap \Phi(g)({}_b\Phi_-)\big) \in \wp({}_a\Lambda). \tag{5.5.4} $$

It follows immediately that this defines a G-cocycle N for $\wp_G(\Lambda)$, so (G, Λ, N) is a set protorootoid. Set $\mathfrak{L}(G, \Phi) = (G, \Lambda, N)$. The map $\mathfrak{L}$ so defined on objects extends naturally to a functor $\mathfrak{L}$ with the desired properties. $\square$

Remarks. (1) In a signed groupoid-set (G, Φ), replacing each set ${}_a\Phi_+$ of $\{\pm\}$ orbit representatives by another has the effect of replacing the cocycle in the corresponding set-protorootoid by another in the same cohomology class. In general, there is no close relation between the corresponding weak orders; if the groupoid is connected and simply connected, this is clear from Lemma 1.14.

(2) In a similar manner as one defines **Gpd-Set**$_\pm$, one can define a category **Gpd-BAlg**$_\pm$ of groupoid representations in suitably defined category of signed Boolean algebras, in which morphisms are natural transformations with positivity-preserving components. For example, for a signed set S, one has a signed Boolean algebra $\wp(S)$ with decomposition $\wp(S) \cong \wp(S_+) \times \wp(S_-)$ into positive and negative Boolean subalgebras corresponding to $S = S_+ \coprod S_-$. The equivalence in the above Proposition may be viewed as a restriction of an equivalence between **Gpd-BAlg**$_\pm$ and **Prd**$_1$.

(3) One may also define similarly a subcategory **Cat-AbCat**$_\pm$ of the category of functors from small categories to a category of small signed ab-categories. The equivalence mentioned in (2) may be viewed as a restriction of an equivalence of **Cat-AbCat**$_\pm$ with a category of triples (G, Λ, N) where G is a small category, Λ is a functor from G to the category of small ab-categories, and N is an idempotent-valued 1-cocycle for a naturally associated non-abelian cohomology theory. Detailed statements and proof of the facts in (2)–(3) lie outside the scope of these papers.

5.6. **Terminology for signed groupoid-sets and set protorootoids.** It is convenient to transfer terminology defined for protorootoids to set protorootoids and signed groupoid-sets in the following ways. Unless otherwise specified, a set protorootoid $T = (G, \Lambda, N)$ will be said to have a property defined for protorootoids if the corresponding protorootoid $\mathfrak{I}(T)$ has that same property. Similarly, a signed groupoid-set $R = (G, \Phi)$ is, unless otherwise specified, said to have such a property if the corresponding protorootoid $\mathfrak{I}\mathfrak{L}(R)$ has that property. In particular, this convention can be applied to define simply connected (connected, complemented, complete, simply or atomically generated, cocycle or interval finite, principal, preprincipal, pseudoprincipal, complete, saturated, rootoidal etc) set protorootoids and signed groupoid-sets, and faithful set protorootoids. Rootoidal set protorootoids are called set rootoids. To avoid confusion with the standard notion of a faithful G-set (e.g. for G a group), a signed groupoid-set (G, Φ) is called strongly faithful if the corresponding protorootoid is faithful.

Similarly, the simple or atomic morphisms (in $\mathrm{mor}(G)$) of T (resp., R) are defined as the simple or atomic morphisms of $\mathfrak{I}(T)$ (resp., $\mathfrak{I}\mathfrak{L}(R)$), etc. This makes sense since $\mathfrak{I}$ and $\mathfrak{L}$ preserve the underlying groupoid.

5.7. For convenience, this subsection explicitly spells out the definition of rootoidal signed groupoid-sets, following the conventions in 5.6, and fixes additional terminology and notation concerning them.

Let $R = (G, \Phi)$ be a signed groupoid-set and $\mathscr{R} = (G, \Lambda, N) := \mathfrak{L}(R)$ denote the corresponding set protorootoid. Use notation as in the proof of Proposition 5.5. For any morphism $g \in {}_aG_b$, define

$$\Phi_g := {}_a\Phi_+ \cap g({}_a\Phi_-). \tag{5.7.1}$$

Note that π_a induces a bijection $\Phi_g \cong N_g$ of sets. The G-cocycle (in fact, coboundary) N' for $\wp_G(\Phi)$ from which N was defined in the proof of 5.5 is, in this notation, $g \mapsto N'(g) = \Phi_g \,\dot\cup\, -\Phi_g$.

The weak order $\{N(g) \mid g \in {}_aG\}$ at a of $\mathscr{R}$ identifies (as poset) with the inclusion ordered set ${}_aL := \{\,\Phi_g \mid g \in {}_aG\,\}$ of subsets of ${}_a\Phi_+$ via the order isomorphism $N_g \mapsto \Phi_g$ for $g \in {}_aG$; this poset ${}_aL$ will be called the weak order of R at a.

It follows directly from the above that R is rootoidal (i.e. by definition, $\mathscr{R}$ is a set rootoid) if and only if the following conditions (i)–(iii) hold:

(i) R is strongly faithful i.e. if $g \in {}_aG$ with $\Phi_g = \emptyset$ then $g = 1_a$.
(ii) For each $a \in \mathrm{ob}(G)$, ${}_aL := \{\,\Phi_g \mid g \in {}_aG\,\}$ is a complete meet semilattice in the order induced by inclusion of subsets of ${}_a\Phi_+$.
(iii) Given $a \in \mathrm{ob}(G)$, a non-empty family $(A_i)_{i\in I}$ in ${}_aL$ and $B \in {}_aL$ such that $A_i \cap B = \emptyset$ for all $i \in I$, if $A := \bigvee_{i\in I} A_i$ exists in ${}_aL$, then $A \cap B = \emptyset$.

5.8. Finally, for convenience of reference, we give the following diagram of functors, which provides considerable latitude in formulation of many results about rootoids and protorootoids.

$$\begin{tikzcd}
\mathbf{Gpd\text{-}Set}_{\pm} \arrow[r, shift left, "\mathfrak{L}"] \arrow[r, phantom, "\cong"] & \mathbf{Set\text{-}Prd} \arrow[l, shift left, "\mathfrak{K}"] \arrow[r, shift left, "\mathfrak{I}'"] \arrow[rr, bend left, "\mathfrak{I}"] & \mathbf{Prd}_1 \arrow[l, shift left, "\mathfrak{J}'"] \arrow[r, shift left, "\mathfrak{I}''"] & \mathbf{Prd} \arrow[l, shift left, "\mathfrak{J}''"] \arrow[ll, bend left, "\mathfrak{J}"] \arrow[r, shift left, "\mathfrak{A}"] \arrow[rr, bend left, "\mathfrak{P}"] & \mathbf{Prd}^a \arrow[l, shift left, "\mathfrak{B}"] \arrow[r, shift left, "\mathfrak{P}'"] & \mathbf{Gpd\text{-}PreOrd} \arrow[l, shift left, "\mathfrak{Q}'"] \arrow[ll, bend left, "\mathfrak{Q}"]
\end{tikzcd}$$

In this, the functors $\mathfrak{L}$, $\mathfrak{I}$, $\mathfrak{A}$, $\mathfrak{P}$, $\mathfrak{B}$ and $\mathfrak{K}$ have been previously defined. The functor $\mathfrak{I}''$ is the evident inclusion functor, and $\mathfrak{I}$ obviously factors via $\mathbf{Prd}_1$ to give a (unique) functor $\mathfrak{I}'$ such that $\mathfrak{I} = \mathfrak{I}''\mathfrak{I}'$. The forgetful functor $\mathfrak{P}$ also obviously factors through abridgement $\mathfrak{A}$ to define the functor $\mathfrak{P}'$ such that $\mathfrak{P} = \mathfrak{P}'\mathfrak{A}$. Since $\mathfrak{A}\mathfrak{B} = \mathrm{Id}$, one has $\mathfrak{P}' := \mathfrak{P}\mathfrak{B}$. The remaining functors $\mathfrak{J}'$, $\mathfrak{J}''$, $\mathfrak{J} = \mathfrak{J}'\mathfrak{J}''$, $\mathfrak{Q}'$ and $\mathfrak{Q} = \mathfrak{B}\mathfrak{Q}'$ (and additional functors involving $\mathbf{Gpd\text{-}PreOrd}_P$ which are relevant to the remarks in 2.16) will be described in subsequent papers. The functors are defined so that each upper functor (with arrow to the right) is right adjoint to the symmetrically corresponding lower functor (with arrow to the left).

6. Examples of rootoids

6.1. **Coxeter systems.** Let S be a set and M be a S-indexed Coxeter matrix. This means that $M = (m_{r,s})_{r,s\in S}$ where $m_{s,r} = m_{r,s} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $r \neq s$ in S

and $m_{r,r} = 1$ for all $r \in S$. The associated Coxeter group W is the group with presentation

(6.1.1) $$W = \langle\, S \mid (rs)^{m_{r,s}} = 1, r, s \in S, m_{r,s} \neq \infty \,\rangle.$$

It is known that the natural map $S \to W$ is an inclusion; we always identify S with a subset of W via this map. It is also known that the order of rs in W is $m_{r,s}$ for any $r, s \in W$. The pair (W, S) is called a Coxeter system (with Coxeter matrix M).

6.2. It is convenient to collect for reference some of the many equivalent characterizations of Coxeter systems. In formulating these, the following general framework will be used.

(∗) (W, S) is a pair consisting of a group W and a set $S \subseteq W$ of involutions (elements of order exactly two) generating W (thus, $1_W \notin S$).

Define the length function $l = l_S \colon W \to \mathbb{N}$ of the pair (W, S) and the subset $T := \{\, wsw^{-1} \mid w \in W, s \in S \,\}$ of W.

6.3. **Exchange condition.** Let (W, S) be a pair satisfying 6.2(∗). Then (W, S) is said to satisfy the *exchange condition* if for all $s_1, \ldots, s_n \in S$ and $s_0 \in S$ with $l(s_0 s_1 \cdots s_n) \leq l(s_1 \cdots s_n) = n$, there is some i with $s_0 s_1 \cdots s_n = s_1 \ldots \hat{s}_i \cdots s_n$, where the term $\hat{s}_i$ is omitted from the product. The *strong exchange condition* is the same except requiring $s_0 \in T$ instead of $s_0 \in S$.

These conditions are often stated in equivalent versions with the hypothesis that $l(s_1 \cdots s_n) = n$ omitted. There are many other variants, including the following two: EC is the condition that for all $w \in W$, $r, s \in S$, with $l(wr) > l(w)$ but $l(swr) \leq l(sw)$, one has $sw = wr$. Further, SEC is the condition that if $w \in W$, $r \in S$, $t \in T$, $l(tw) \geq l(w)$ but $l(twr) \leq l(wr)$ then $tw = wr$. The following is well known.

Proposition. *If (W, S) satisfies 6.2(∗), the following conditions are equivalent:*

(i) (W, S) *is a Coxeter system.*
(ii) (W, S) *satisfies the exchange condition.*
(iii) (W, S) *satisfies the strong exchange condition.*
(iv) (W, S) *satisfies EC.*
(v) (W, S) *satisfies SEC.*

Proof. It is easy to see that (v) implies (iv), (iii) and (ii). The equivalence of (i) and (ii) is in [3]. Finally, (i) implies (v) by a routine computation with the reflection cocycle of (W, S) given in Remark 6.4. □

6.4. **Reflection cocycle.** Let (W, S) satisfy 6.2(∗). Define the W-set

$$T = \{\, wsw^{-1} \mid w \in W, s \in S \,\},$$

with (left) W-action by conjugation $(w, t) \mapsto wtw^{-1}$. Let $\Lambda \colon W \to \mathbf{Set}$ denote the functor corresponding to the W-set T (regarding W as one-object groupoid). This gives a functor $\wp_W(\Lambda) \colon W \to \mathbf{BRng}$, which affords the conjugacy representation of W on $\wp(T)$. From [10], one has the following.

Proposition. *The pair* (W,S) *is a Coxeter system if and only if there is a cocycle* $N\colon W \to \wp_W(\Lambda)$ *such that* $N(r)=\{r\}$ *for all* $r\in S$. *In that case, for all* $w\in W$, *one has* $N(w)=\{\, t\in T \mid l(tw)<l(w)\,\}$, $l(tw)\equiv l(w)+1 \pmod 2$ *for all* $t\in T$ *and* $l_S(w)=|N(w)|$.

If (W,S) is a Coxeter system, T is called the set of *reflections* and N is called the *reflection cocycle* of (W,S). One then has a set protorootoid $\mathscr{C}'_{(W,S)} := (W,\Lambda,N)$ and its corresponding protorootoid $\mathscr{C}_{(W,S)} := \mathfrak{I}(\mathscr{C}'_{(W,S)}) = (W,\wp_W(\Lambda),N)$.

Remarks. To illustrate the usefulness of the Proposition for computations, we use it to show that a Coxeter system (W,S) satisfies SEC. Suppose $l(tw)\geq l(w)$ but $l(twr)\leq l(wr)$ where $t\in T$, $r\in S$. Then $t\notin N(w)$ but

$$t\in N(wr)=N(w)+wN(r)w^{-1}=N(w)+\{wrw^{-1}\},$$

so $t=wrw^{-1}$ and $tw=wr$. This argument also shows that $l(wr)=l(w)+1$.

6.5. **Root system.** Using the generalities in Section 5, another equivalent reformulation of the last characterization of Coxeter systems is as follows.

Proposition. *The pair* (W,S) *satisfying* 6.2$(*)$ *is a Coxeter system if and only if there is an action of* W *on the set* $T\times\{\pm\}$ *such that for* $s\in S$, $t\in T$, $\epsilon\in\{\pm\}$ *one has*

$$s(t,\epsilon)=(sts,\nu\epsilon),\qquad \nu=\begin{cases}+, & \text{if } s=t\\ -, & \text{if } s\neq t.\end{cases}\tag{6.5.1}$$

It is known from [3] (and follows also from the Proposition, Proposition 6.4 and Section 5) that if this action exists, it is given explicitly by

$$w(t,\epsilon)=(wtw^{-1},\nu\epsilon),\qquad \nu=\begin{cases}+, & \text{if } l(wt)>l(w)\\ -, & \text{if } l(wt)<l(w).\end{cases}\tag{6.5.2}$$

For any Coxeter system (W,S), this gives a a $(W\times\{\pm\})$-set $\Phi := T\times\{\pm\}$, called the *abstract root system* of (W,S), with $\{\pm\}$ action by multiplication on the right factor and positive roots $\Phi_+ := T\times\{+\}$. For $\alpha=(t,\epsilon)\in\Phi$, define the corresponding reflection $s_\alpha := t\in T$. One then has

$$N(w)=\{\, s_\alpha \mid \alpha\in\Phi_w\,\},\qquad \Phi_w := \Phi_+\cap w(-\Phi_+).\tag{6.5.3}$$

Viewing W as one-object groupoid and Φ as a functor $\Phi\colon W\to \mathbf{Set}_\pm$, define $C_{(W,S)} := (W,\Phi)$ regarded as object of $\mathbf{Gpd\text{-}Set}_\pm$ and call it the *standard signed groupoid-set* of (W,S). Comparing the above with Section 5 shows that $\mathfrak{L}(\mathscr{C}_{(W,S)})\cong C_{(W,S)}$.

The set $\Pi := \{\, \alpha\in\Phi_+ \mid s_\alpha\in S\,\}$ is called the set of *simple roots* of Φ. For any $J\subseteq S$, the subgroup $W_J := \langle\, J\,\rangle$ is called the *standard parabolic subgroup* of W generated by J and $\Pi_J := \{\, \alpha\in\Pi \mid s_\alpha\in J\,\}$ is called the set of simple roots of W_J.

6.6. This subsection contains informal remarks about other, more standard, notions of root systems of Coxeter groups. It is well-known that Coxeter systems (W, S) admit geometric realizations as reflection groups associated to root systems $\Psi \subseteq V$ in real vector spaces V. Such root systems will be loosely referred to as *linearly realized* root systems. Consider those in [16, Ch 5], for example. Note that Ψ is *reduced* in the sense that the only scalar multiples of a root α which are roots are $\pm\alpha$. Regard Ψ as a W-set with commuting free action of the sign group $\{\pm\}$ by multiplication by ± 1, and the standard system of positive roots Ψ_+ as orbit representatives. For each $\alpha \in \Psi$, there is an associated reflection $r_\alpha \in T \subseteq W$. There is an isomorphism of $W \times \{\pm\}$-sets $\theta\colon \Psi \mapsto \Phi = T \times \{\pm\}$ given by $(\alpha, \epsilon) \mapsto (r_\alpha, \epsilon)$ for $\alpha \in \Psi_+$, $\epsilon \in \{\pm\}$; this holds since the W-action on $T \times \{\pm\}$ is determined by (6.5.1) and there is an analogous formula for the W-action on Φ (which is easily checked using the well-known fact that the only positive root of Ψ made negative by a simple reflection is its corresponding simple root). The isomorphism maps Ψ_+ bijectively to Φ_+ and satisfies $r_{\theta(\alpha)} = s_\alpha$ for $\alpha \in \Psi$.

One may view Ψ as a functor and (W, Ψ) as object of $\mathbf{Gpd\text{-}Set}_\pm$; as such, it is clearly isomorphic to $C_{(W,S)} = (W, \Phi)$. Similar remarks can be made for many other classes of linearly realized root systems of Coxeter groups which are reduced in the above sense and *real* in that they do not contain imaginary roots as in the root systems of Kac-Moody Lie algebras. In the case of a finite Weyl group W, for example, one obtains a signed groupoid-set (W, Ψ) isomorphic to (W, Φ) from a reduced (crystallographic) root system Ψ of (W, S) and a positive system Ψ_+, in the sense of [3]. More generally still, the real compressions (as informally described in the introduction, and defined in subsequent papers) of linearly realized root systems of Coxeter groups in the literature are isomorphic to (W, Φ), whether they are real and reduced or not.

Many of the most important applications of Coxeter groups involve natural occurrences of linearly realized root systems (e.g. in the theory of semisimple complex Lie algebras) and existence of such root systems in general provides powerful techniques for the deeper study of Coxeter groups and related structures. One expects that root systems in the abstract sense of these papers will not have linear realizations in real vector spaces in a similar sense in general (in the case of complete, principal, rootoidal signed groupoids-sets attached to non-realizable simplicial oriented geometries, for example). Some results concerning preservation of realizability under the main constructions of these papers will be given in subsequent papers.

6.7. **Standard rootoid of a Coxeter system.** The standard protorootoids attached to Coxeter systems are the motivating examples of principal rootoids.

Theorem. *Let (W, S) be a Coxeter system.*

(a) *The triple $\mathscr{C} = \mathscr{C}_{(W,S)} := (W, \wp_W(\Lambda), N)$ is a principal rootoid with S as its simple generators. It is complete if and only if W is finite.*
(b) *$\mathscr{C}' = \mathscr{C}'_{(W,S)} := (W, \Lambda, N)$ is a (principal) set rootoid.*
(c) *The (principal, rootoidal) signed groupoid-set $\mathfrak{R}(\mathscr{C}'_{(W,S)})$ is isomorphic to the standard signed groupoid-set $C_{(W,S)} := (W, \Phi)$ of (W, S).*

Proof. It has already been noted that $\mathscr{C}' = (W, \Lambda, N)$ is a set protorootoid, that $\mathscr{C} := \mathfrak{I}(\mathscr{C}') = (W, \wp(\Lambda), N)$ is its corresponding protorootoid and that $C_{(W,S)} \cong \mathfrak{K}(\mathscr{C}')$ is the corresponding signed groupoid-set. So only (a) requires proof. By Proposition 6.4,

$$l(w) = |N(w)|, \text{ for } w \in W \tag{6.7.1}$$

where $l := l_S$. Therefore, $|N(w)| = 0$ if and only if $l(w) = 0$ if and only if $w = 1_W$, since S generates W. This implies that $\mathscr{C}$ is faithful i.e. satisfies 4.3(i). Note that the elements of the Boolean ring $\wp(T)$ of finite rank are the finite subsets, and their rank is their cardinality. Therefore, (6.7.1) implies that S is the set of simple morphisms of W (so $\mathscr{C}$ is simply generated) and that $l_S = l_N$. This shows that $\mathscr{C}$ is a faithful, principal protorootoid. To show that $\mathscr{C}$ is a rootoid, it remains to prove that $\mathscr{C}$ satisfies 4.3(ii)–(iii).

The weak right order $\leq$ of $\mathscr{C}$ at the unique object of W (as groupoid) is, by definition, the partial order $\leq$ of W defined by $x \leq y$ if and only if $N(x) \subseteq N(y)$; this is consistent with the usual definition of right weak order of W by [1, Proposition 3.1.3] or Corollary 3.14. It is well known that $(W, \leq)$ is a complete meet semilattice (see [1, Theorem 3.2.1]) i.e. 4.3(ii) holds. The JOP (property 4.3(iii)) is not standard, but it (and also 4.3(ii)) is proved in [9]. Hence $\mathscr{C}$ is a rootoid. By definition, $\mathscr{C}$ is complete if and only if W has a maximal element in weak right order. It is well known that such a maximal element exists if and only if W is finite (in which case, it is the longest element of W with respect to l). This completes the proof. $\square$

Remarks. Given a Coxeter groupoid G with a (linearly realized, real) root system Φ (as defined in [14] and [6]), one can show similarly that, regarding Φ just as signed G-set in the natural way, (G, Φ) is a principal, rootoidal signed groupoid-set. (In fact, [13] proves that the weak orders of the subclass of (finite) Weyl groupoids have complete ortholattices as their weak orders.) Also, (G, Φ) is complete if and only if each component of G is finite. One proof uses results from [14] and [6] along with extensions to Coxeter groupoids of some of the results in [9] (note that no extensions of the arguments in [9] involving Bruhat order of W are known). Another proof of this fact can be given along the lines of 6.10. A proof of a more general fact will be given in subsequent papers.

6.8. **Reflection subgroups.** It is worth observing that inclusions of reflection subgroups of Coxeter groups give rise to morphisms in $\mathbf{Prd}'$, instead of in the category $\mathbf{Prd}$ with which this series of papers is more concerned. To see this, let (W, S) be a Coxeter system, $\mathscr{R} = (W, \Lambda, N) := \mathscr{C}'_{(W,S)}$ and W' be a reflection subgroup of W i.e. a subgroup W' of W such that $W' = \langle T' \rangle$ where $T' := T \cap W'$. Let Λ' be the natural conjugacy representation of W' on T'. Let $i\colon W' \to W$ and $j\colon T' \to T$ denote the inclusion maps. There is a natural transformation $\nu\colon \Lambda' \to \Lambda i$ with j as its unique component. Then $\wp_{W'}(\nu)\colon \wp_W(\Lambda)i \to \wp_{W'}(\Lambda')$ is a natural transformation with unique component $\wp(j)$ where $\wp(j)(A) := j^{-1}(A) = A \cap T'$ for all $A \subseteq T$. Let $N'\colon W \to \wp(T')$ be the composite $N' := \wp(j)Ni$ i.e. $N'(w) := N(w) \cap T'$ for $w \in W'$. The definitions imply that N' is a W'-cocycle for $\wp_{W'}(\Lambda')$ i.e. $N' \in$

$Z^1(W', \wp_{W'}(\Lambda'))$. Hence $\mathscr{R}' := (W', \Lambda', N')$ is a set protorootoid. It is easily checked that $(i, \wp_{W'}(\nu))\colon (W', \wp_{W'}(\Lambda'), N') \to (W, \wp_W(\Lambda), N)$ is a morphism in $\mathbf{Prd}'$.

The above-mentioned facts are all purely formal consequences of the definitions. According to [10], there is a subset S' of W' such that (W', S') is a Coxeter system and $\mathscr{R}' = \mathscr{C}_{(W',S')}$ (necessarily, $S' = \{\, s \in W' \mid |N'(s)| = 1 \,\}$).

Remarks. From the above formula $N'(w) := N(w) \cap T'$ for $w \in W'$, it follows that the identity map on W' is an order preserving map from W', ordered by the restriction to W' of weak order on W, to W' in its weak order. The map is not in general an order isomorphism.

6.9. **Semilocal criterion for rootoids.** The following result is closely related to a criterion in [2] for a finite poset to be a lattice, and its proof is very similar in arrangement to the argument from [9, Section 4] used to establish the JOP in the proof of Theorem 6.7.

Proposition. *Assume that $\mathscr{R} = (G, \Lambda, N)$ is an interval finite, faithful protorootoid. Let $A := A_{\mathscr{R}}$ denote its set of atoms. Then $\mathscr{R}$ is a rootoid if and only if it satisfies the following condition: whenever $a \in \mathrm{ob}(G)$, $r, s \in {}_aA$ and $g \in {}_aG$ are such that $N_r \cap N_g = N_s \cap N_g = \emptyset$ and N_r, N_s have an upper bound in ${}_a\mathscr{L}$, then the join $N_r \vee N_s$ exists in ${}_a\mathscr{L}$ and satisfies $(N_r \vee N_s) \cap N_g = \emptyset$.*

Remarks. The condition in the proposition will be called the semilocal criterion (SLC). In this, the term "semilocal" is intended to suggest that the condition is not entirely local, in the sense of being expressed purely in terms of the generators, but involves also general elements of the groupoid. No corresponding general local criterion is known.

Proof. Make the assumptions of the proposition. Then A generates G, by Lemma 3.5. Validity of SLC is clearly necessary for $\mathscr{R}$ to be a rootoid. Conversely, suppose that the condition holds. The following statement will be proved:

(∗) If $a \in \mathrm{ob}(G)$ and $x, y, g \in {}_aG$ are such that $N_x \cap N_g = N_y \cap N_g = \emptyset$ and N_x, N_y have an upper bound N_u in ${}_a\mathscr{L}$, then the join $N_x \vee N_y$ exists in ${}_a\mathscr{L}$ and satisfies $(N_x \vee N_y) \cap N_g = \emptyset$.

Let us first check that (∗) would imply that $\mathscr{R}$ is a rootoid. Note that (taking $g = 1_a$), (∗) implies that if elements x, y of ${}_a\leq$ have an upper bound, they have a least upper bound. This in turn implies that ${}_a\leq$ is a complete meet semilattice. For given a non-empty subset X of ${}_aG$, with say $x \in X$, $\bigwedge X = \bigvee Y$ where Y is the set of lower bounds of X; the join of Y exists from the above since Y is bounded above by x (and hence in particular is finite by interval finiteness of $\mathscr{R}$). This shows that (∗) implies that $\mathscr{R}$ satisfies 4.3(ii). Note also that if a subset X of ${}_aG$ is bounded above, then, writing $X = \{x_1, \ldots, x_n\}$, one has $\vee_i x_i = x_1 \vee (x_2 \vee (x_3 \vee \ldots (x_{n-1} \vee x_n)))$. From this, it follows inductively that (∗) implies that $\mathscr{R}$ satisfies JOP, and hence is a rootoid.

Now we prove (∗) by induction on the cardinality $n(u) := |[1_a, u]|$ of the interval $[1_a, u]$ in ${}_a\leq$. The statement is trivial if $u = 1_a$ or $x = 1_a$ or $y = 1_a$. Assume inductively that the statement holds if $n(u) < m$. Suppose that $n(u) = m$, $x \neq 1_a$

and $y \neq 1_a$. Choose $r, s \in {}_aA$ so $1_a < r \leq x$ and $1_a < s \leq y$. Since u is an upper bound of r, s in ${}_aG$, $z := r \vee s$ exists and satisfies $z \leq u$, $N_z \cap N_g = \emptyset$.

Since $s \leq z$, we have $N_z = N_s \dot\cup s(N_{s^*z})$. Since $N_s \cap N_g \subseteq N_y \cap N_g = \emptyset$, we have $N_{s^*g} = N_{s^*} \dot\cup s^*(N_g)$. It follows that $N_{s^*z} \cap N_{s^*g} = \emptyset$. Similarly, $N_{s^*y} \cap N_{s^*g} = \emptyset$. Let $s \in {}_aA_b$. Now we use Lemma 4.4. In weak order on ${}_bG$, s^*u is an upper bound of s^*y and s^*z. Also, $[1_b, s^*u] = s^*[s, u] \subsetneq s^*[1, u]$ so $n(s^*u) < n(u)$. By induction, there is $w \in {}_aG$ such that $s^*z \vee s^*y = s^*w$ and $N_{s^*w} \cap N_{s^*g} = \emptyset$. In particular, $N_{s^*w} \cap N_{s^*} = \emptyset$, so $N_w = N_s \dot\cup s(N_{s^*w})$. It follows by Lemma 4.4 that $z \vee y = w$ in ${}_aG$ (since the join $z \vee y$ is the same whether calculated in ${}_aG$ or in the principal order coideal of ${}_aG$ generated by s). Also, it is easily seen from above that $N_w \cap N_g = \emptyset$.

A similar argument to that in the last paragraph, with s replaced by r, z replaced by w, and y replaced by x, shows that $v := w \vee x$ exists in ${}_aG$ and satisfies $N_v \cap N_g = \emptyset$. The proof is completed by noting that

$$v = x \vee w = x \vee (z \vee y) = x \vee (r \vee s) \vee y = (x \vee r) \vee (s \vee y) = x \vee y. \qquad \square$$

6.10. **Another proof of Theorem 6.7.** The SLC can be used to give an alternative proof that $\mathscr{C}_{(W,S)}$ is a rootoid, with the advantage of being self-contained except for well-known properties of shortest coset representatives of (rank two) standard parabolic subgroups. The argument at the beginning of the proof of Theorem 6.7 implies that $\mathscr{C}$ is faithful and principal. It is interval finite by Lemma 3.6, and $A = S$ by Lemma 3.7. As in the proof of Theorem 6.7, it is possible to use the description of weak right order of $\mathscr{C}$ in terms of l given by Corollary 3.14.

To show that $\mathscr{C}$ is a rootoid, it will suffice to verify the SLC. In terms of l, SLC requires that if $r, s \in S$ and $g \in W$ with $l(r^{-1}g) > l(g)$ and $l(s^{-1}g) > l(g)$ and r, s have have an upper bound x in weak right order on W, then the join $y = r \vee s$ exists in weak right order on W and satisfies $l(y^{-1}g) = l(y) + l(g)$. Now $l(r^{-1}x) < l(x)$ and $l(s^{-1}x) < l(x)$. Let $x' \in W_Jx$, where $J := \{r, s\}$, be the shortest coset representative in W_Jx i.e. $x' \in W_Jx$ and $l(yx') = l(y) + l(x')$ for all $y \in W_J$. Write $x = yx'$ where $y \in W_J$. Then $l(s^{-1}y) < l(y)$ and $l(r^{-1}y) < l(y)$, which implies that W_J is finite and y is the longest element of W_J. It is easy to check that $y = r \vee s$. The assumptions imply that g is the shortest coset representative in W_Jg. Hence $l(y^{-1}g) = l(y) + l(g)$ as required. This implies that $\mathscr{C}$ is a principal rootoid, with simple generators S. The proof of the criterion in Theorem 6.7 for completeness of $\mathscr{C}$ is as in the earlier proof.

6.11. **Rootoids from simplicial hyperplane arrangements.** Let V be a real Euclidean space with inner product $\langle -, - \rangle$. An arrangement $\mathscr{H}$ of hyperplanes in V is a set of affine hyperplanes in V. (For background on hyperplane arrangements needed here, a convenient source is [2], which we shall follow as regards terminology and notation since the main result of this subsection is essentially a direct translation into protorootoid terminology of a main result of that paper) The arrangement $\mathscr{H}$ is said to be central if all the hyperplanes in $\mathscr{H}$ are linear hyperplanes (i.e contain 0) and to be essential if their normals span V.

Fix a finite, essential, central hyperplane arrangement $\mathscr{H}$ in V, with $V \neq 0$ to avoid trivialities. The connected components of $V \setminus \cup_{H \in \mathscr{H}} H$ (in the metric topology

induced by the inner product $\langle -, - \rangle$) are called the chambers of $\mathscr{H}$. One says that two chambers D, E are separated by the hyperplane $H \in \mathscr{H}$ if they are in different connected components of $V \setminus H$. Two chambers are said to be adjacent if they are separated by a unique hyperplane $H \in \mathscr{H}$.

There are finitely many chambers. Consider the (unique up to isomorphism) connected, simply connected groupoid G with the chambers of $\mathscr{H}$ as its objects. For two chambers D, E, let $f_{E,D} \colon D \to E$ be the unique morphism in G from D to E.

Define an object $R = C_{\mathscr{H}} := (G, \Phi)$ of $\mathbf{Gpd}\text{-}\mathbf{Set}_{\pm}$ as follows. Let U be the set of all unit normal vectors to hyperplanes $H \in \mathscr{H}$, regarded as indefinitely signed set with action of the sign group $\{\pm\}$ by multiplication. Next, define a definitely signed set ${}_D\Phi$ for each chamber D, with underlying indefinitely signed set U and with

$$ {}_D\Phi_+ := \{\, u \in {}_D\Phi \mid \langle u, D \rangle \subseteq \mathbb{R}_{>0} \,\}. \tag{6.11.1} $$

The functor $\Phi \colon G \to \mathbf{Set}_{\pm}$ is defined by setting $\Phi(D) := {}_D\Phi$ as definitely signed set and requiring that the composite of Φ with the forgetful functor $\mathbf{Set}_{\pm} \to \mathbf{Set}_{\{\pm\}}$ be equal to the constant functor $G \to \mathbf{Set}_{\{\pm\}}$ with value U. This completes the definition of $R = C_{\mathscr{H}}$. Define the associated set protorootoid $\mathscr{C}'_{\mathscr{H}} := \mathfrak{L}(R)$ and protorootoid $\mathscr{C}_{\mathscr{H}} := \mathfrak{I}(\mathfrak{L}(R))$

Theorem. *The signed groupoid-set $R = C_{\mathscr{H}}$ of a finite, central, essential hyperplane arrangement $\mathscr{H}$ in the real Euclidean space V is rootoidal if and only if $\mathscr{H}$ is a simplicial arrangement i.e. if and only if each chamber is an open simplicial cone in V. In that case, R is complete and principal and the simple generators of the underlying groupoid are the morphisms between adjacent chambers.*

Proof. For $E, D \in \Phi$, $\Phi_{f_{E,D}}$ is the set of all unit normals in ${}_E\Phi_+$ to hyperplanes in $\mathscr{H}$ which separate E and D. Thus, the cardinality $|\Phi_{f_{E,D}}|$ is the number of hyperplanes in $\mathscr{H}$ separating D and D, which is called the *combinatorial distance* between E and D. The combinatorial distance is 0 if and only if $E = D$, so R is strongly faithful i.e R satisfies 5.7(i). Now $D \;{}_E{\leq}\; D'$ in the right weak order ${}_E{\leq}$ on ${}_EG$ if and only if every hyperplane in $\mathscr{H}$ separating E and D also separates E and D'. Thus, the right weak order ${}_E{\leq}$ on ${}_EG$ is just the poset of regions (chambers) of $\mathscr{H}$ ordered by combinatorial distance from the base chamber E, as studied in [2].

Let S be the set of all simple morphisms in $\mathrm{mor}(G)$ i.e. the set of all $f_{E,D}$ such that $|\Phi_{f_{E,D}}| = 1$. Restating basic facts from [2] in the terminology here, S is a set of groupoid generators of G and

$$ |\Phi_g| = l_S(g), \text{ for } g \in \widehat{G}. \tag{6.11.2} $$

Hence R is a principal signed groupoid-set in general.

For regions E, D of $\mathscr{H}$, note that $-D$ is a region and

$$ \Phi_{f_{E,-D}} = {}_E\Phi_+ \setminus \Phi_{f_{E,D}}. \tag{6.11.3} $$

To prove that R satisfies the JOP (i.e. 4.3(iii)), it is necessary to show that if E, A_i and A are chambers such that $\Phi_{f_{E,A_i}} \cap \Phi_{f_{E,A}} = \emptyset$ for all i and $f_{E,B} := \bigvee_i f_{E,A_i}$ exists in $({}_EG,\ {}_E{\leq})$ then $\Phi_{f_{E,B}} \cap \Phi_{f_{E,A}} = \emptyset$. But under those assumptions, $\Phi_{f_{E,A_i}} \subseteq \Phi_{f_{E,-A}}$

for all i so by definition of join, $\Phi_{f_{E,B}} \subseteq \Phi_{f_{E,-A}}$ and therefore $\Phi_{f_{E,B}} \cap \Phi_{f_{E,A}} = \emptyset$ (cf. Lemma 4.1).

From the above, it now follows that R is rootoidal if and only if it satisfies 4.3(ii) or equivalently if and only if for every chamber, the poset of regions of $\mathscr{H}$ oriented from that chamber is a complete semilattice. The weak order ${}_E{\leq}$ at E is finite, and clearly has a maximum element $-E$, so it is a complete meet semilattice if and only if it is a lattice. By [2], ${}_E{\leq}$ is a lattice for all E if and only if $\mathscr{H}$ is simplicial. The argument shows that if R is rootoidal, it is complete and principal. $\square$

Remarks. Let W of a finite Coxeter group W acting (with no pointwise fixed subspace of positive dimension) as reflection group on a Euclidean space. Consider the reflection arrangement $\mathscr{H}$, which consists of the reflecting hyperplanes of W. This arrangement is well known to be simplicial. Fix a fundamental chamber C of the arrangement; the set S of reflections of W in walls of C makes (W,S) a Coxeter system. The simply connected groupoid underlying $R_{\mathscr{H}}$ is canonically isomorphic to the universal covering groupoid of W (using the natural bijection between chambers and elements of W afforded by the choice of C). One can check that this groupoid isomorphism underlies an isomorphism of the rootoid $\mathscr{C}_{\mathscr{H}}$ attached to $\mathscr{H}$ with the universal covering rootoid of $\mathscr{C}_{(W,S)}$.

6.12. **Other examples of rootoids.** To partly offset any misleading impression due to the fact that the preceding examples were all principal rootoids, this subsection describes some non-principal rootoids of a quite different character.

Example. (1) Let G denote the additive group $\mathbb{R}$ with standard partial order. Let P denote the cone of non-negative elements $P := \{\, x \in G \mid x \geq 0 \,\}$. Denote addition in G by $+$ and addition (symmetric difference) in the Boolean ring $\wp(G)$ by $\dotplus$ to avoid confusion. Attached to (G,P) there is a protorootoid $\mathscr{R} := (G, \Lambda, N)$ as follows. The functor $\Lambda\colon G \to \mathbf{BRng}$ gives the translation action of G on subsets of G; formally, Λ sends the unique object of G to $\wp(G)$ and

$$\Lambda(\lambda)(A) := A + \lambda = \{\, a + \lambda \mid a \in A \,\}$$

for any morphism $\lambda \in G$ and any $A \in \wp(G)$. The cocycle N is the coboundary $N \in B^1(G,\Lambda)$ defined by $N(\lambda) = (\lambda + P) \dotplus P$. Note that either $N(\lambda)$ is empty, or it is a closed-open interval in $\mathbb{R}$ with 0 as either the (closed) left endpoint or (open) right endpoint. It is straightforward to check that $\mathscr{R}$ is a rootoid, which is not interval finite. The unique weak right order is the partial order $\preceq$ of $\mathbb{R}$ such that $\lambda \preceq \mu$ if $\lambda\mu \geq 0$ and $|\lambda| \leq |\mu|$, where $|\nu|$ denotes the absolute value of $\nu \in \mathbb{R}$.

(2) Let V be a real Euclidean space. Let $G = \mathrm{O}(V)$ denote the corresponding real orthogonal group. A vector space total ordering of V is given by the lexicographic ordering on coordinate vectors with respect to a chosen ordered orthonormal basis of V. The natural action of G on the set of vector space total orderings $\leq$ of V is simply transitive. Fix such an ordering $\leq$. Let $\mathbb{S} := \{\, v \in V \mid \langle v, v \rangle = 1 \,\}$ denote the unit sphere in V and $\mathbb{S}_+ := \{\, v \in \mathbb{S} \mid v > 0 \,\}$. This makes $\mathbb{S}$ into a definitely signed set. The group G and sign group $\{\pm 1\}$ have natural commuting actions on $\mathbb{S}$. Let $\Phi\colon G \to \mathbf{Set}_{\pm}$ be the representation of G taking the value $\mathbb{S}$ at the unique object of G, and with the natural action of G on the unit sphere as underlying

representation of G in **Set**. This defines a signed groupoid-set $R = C_{\mathrm{O}(V)} := (G, \Phi)$. Let $\mathscr{C}_{\mathrm{O}(V)} := \mathfrak{L}(R)$ and $\mathscr{C}'_{\mathrm{O}(V)} := \mathfrak{I}(\mathfrak{L}(R))$ denote the associated set protorootoid and protorootoid, respectively. It is shown in [20] that $C_{\mathrm{O}(V)}$ is complete and rootoidal.

For example, take $V = \mathbb{R}^2$ with dot product. Then $\mathbb{S} = \{ (x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$ and one can take (for instance)

$$\mathbb{S}_+ := \{ (x, y) \in \mathbb{S} \mid y > 0 \text{ or } (y = 0 \text{ and } x > 0) \}.$$

Define a bijection $f\colon [0, \pi) \to \mathbb{S}_+$ by $f(t) = (\cos(t), \sin(t))$. The sets Φ_g for $g \in \mathrm{O}(V)$ are $\emptyset$, $\mathbb{S}_+$, $\{(1, 0)\}$, $\mathbb{S}_+ \setminus \{(1, 0)\}$ and the (open, closed or open-closed) arcs $f(I)$ where I is a subinterval of $[0, \pi)$ of the form $[0, t)$, $[0, t]$, (t, π) or $[t, \pi)$ for some $t \in (0, \pi)$. It is easy to check from this that $C_{\mathrm{O}(V)}$ is a complete, regular, saturated, pseudo-principal, rootoidal signed groupoid-set. Clearly $C_{\mathrm{O}(V)}$ is not interval finite. The orthogonal reflection which fixes the y-axis pointwise is the unique simple morphism.

7. Complete rootoids

7.1. The following proposition gives analogues of standard properties of longest elements of finite Coxeter systems. Note that the abridgement of any faithful, complete protorootoid satisfies the hypotheses.

Proposition. *Suppose that $\mathscr{R} = (G, \Lambda, N)$ is a faithful, complete, abridged protorootoid. Let $a \in \mathrm{ob}(G)$.*

- (a) *The weak order ${}_a{\le}$ has a maximum element $\omega(a)$. It satisfies $\omega(a) = \bigvee {}_aG$ in ${}_aG$ and $N(\omega(a)) = \bigvee_a \mathscr{L}$ in ${}_a\mathscr{L}$.*
- (b) *$e_a := N(\omega(a))$ is an identity element of (the Boolean algebra) ${}_a\Lambda$.*
- (c) *$\mathscr{R}$ is a unitary protorootoid.*
- (d) *Let $a' := \mathrm{dom}(\omega(a)) \in \mathrm{ob}(G)$, so that $\omega(a) \in {}_aG_{a'}$. Then $\omega(a)^* = \omega(a')$.*
- (e) *The map $h \mapsto \omega(a)h\colon {}_{a'}G \to {}_aG$ is an order anti-isomorphism in the weak right orders.*
- (f) *For $g \in {}_bG_a$, $N(g\omega(a)) = N(g)^{\complement} := e_b + N(g)$ in ${}_b\Lambda$.*
- (g) *${}_a\mathscr{L}$ is a complete ortholattice with orthocomplement given by restriction of complement $A \mapsto A^{\complement}$ in the Boolean algebra ${}_a\Lambda$.*
- (h) *$\mathscr{R}$ is a complemented, complete rootoid.*

Proof. Note that ${}_a\mathscr{L}$ has a maximum element by definition, because $\mathscr{R}$ is complete. Since $\mathscr{R}$ is faithful, there is an order isomorphism $x \mapsto N(x)\colon {}_a\mathscr{L} \cong {}_aG$, and the rest of (a) follows trivially. From (a), for all $g \in {}_aG$ one has $N(g) \subseteq N(\omega(a))$ so $N(g) \cap N(\omega(a)) = N(g)$. Since $\mathscr{R}$ is abridged, ${}_a\Lambda$ is generated as ring by ${}_a\mathscr{L}$ and (b) follows.

For any $g \in {}_aG_b$, $\Lambda(g)\colon \Lambda(b) \to \Lambda(a)$ is a homomorphism of Boolean rings. Hence it is order preserving. Since it is bijective, with inverse $\Lambda(g^*)$, it must preserve maximum elements i.e. it is a homomorphism of Boolean algebras. This proves (c).

Next, observe that by (b) and (c),

$$N(\omega(a)\omega(a')) = N(\omega(a)) + \omega(a)N(\omega(a')) = e_a + \omega(a)(e_{a'}) = e_a + e_a = 0 = N(1_a).$$

Since $\mathscr{R}$ is faithful, $\omega(a)\omega(a') = 1_a$ and (d) follows.

For any $h \in {}_{a'}G$, one has $N(\omega(a)h) = N(\omega(a)) + \omega(a)(N(h)) = \big(\omega(a)N(h)\big)^{\complement}$. This immediately implies that the map in (e) is weak right order reversing. Then (e) follows since, by symmetry, an inverse order anti-isomorphism is given by $k \mapsto \omega(a')k\colon {}_aG \to {}_{a'}G'$.

For g as in (f), the cocycle condition gives

$$N(g\omega(a)) = N(g) + gN(\omega(a)) = N(g) + g(e_a) = N(g) + e_b = N(g)^{\complement}$$

which proves (f). For $A \in {}_b\mathscr{L}$, write $A = N(g)$ where $g \in {}_bG_a$. Then by (f), $A^{\complement} = N(g\omega(a)) \in {}_b\mathscr{L}$. This easily implies (g). For (h), note that $({}_a\Lambda, {}_a\mathscr{L})$ is a complemented protomesh, by (g). Therefore, JOP follows by Lemma 4.1(a) and $\mathscr{R}$ is a rootoid. Also by (g), $\mathscr{R}$ is complete and complemented, proving (h). $\square$

7.2. Proposition 7.1 applies to the abridgement $\mathscr{R}^a$ of any complete rootoid $\mathscr{R}$. The next corollary describes, for such $\mathscr{R}$, the analogue of the automorphism of a Coxeter system defined by conjugation by the longest element.

Corollary. *Let $\mathscr{R} = (G, \Lambda, N)$ be a complete rootoid.*

(a) *There is a unique functor $d\colon G \to G$ such that there is natural isomorphism $\omega\colon d \to \mathrm{Id}_G$ with component $\omega_a = \omega(a)$ at each $a \in \mathrm{ob}(G)$.*
(b) *Let $D := (d, \Lambda\omega^*)$. Then $D \in \mathrm{Aut}_{\mathbf{RdE}}(\mathscr{R})$ and $D^2 = \mathrm{Id}_{\mathscr{R}}$.*

Proof. For $a \in \mathrm{ob}(G)$, let $d(a) := a' = \mathrm{dom}(\omega(a))$ where $\omega(a), a'$ are as defined in Proposition 7.1 for $\mathscr{R}^a$. For $f \in {}_aG_b$, define $d(f) \in {}_{d(a)}G_{d(b)}$ by $d(f) := \omega(a)^* f \omega(b)$. It is easy to check that this defines the unique functor as required in (a). By Remark 4.9, it is sufficient to check (b) with **RdE** replaced by **Prd**. This is done by a straightforward calculation using the definitions and Proposition 7.1. $\square$

7.3. A protorootoid $\mathscr{R} = (G, \Lambda, N)$ is said to be *finite* if $\mathrm{mor}(G)$ is finite (and hence $\mathrm{ob}(G)$ is finite) and ${}_a\Lambda$ is a finite Boolean ring for all $a \in \mathrm{ob}(G)$.

Corollary. *Let $\mathscr{R} = (G, \Lambda, N)$ be a rootoid.*

(a) *If $\mathscr{R}$ is interval finite, complete and connected, then G is finite.*
(b) *If $\mathscr{R}$ is cocycle finite, complete, abridged and connected, then it is finite.*
(c) *If $\mathscr{R}$ is principal, complete, abridged and connected, then it is finite.*

Proof. If $\mathrm{ob}(G) = \emptyset$, this is trivial, so assume that $\mathrm{ob}(G) \neq \emptyset$. Let $a \in \mathrm{ob}(G)$ and assumptions be as in (a). Then ${}_aG = [1_a, \omega(a)]_{{}_a\mathscr{L}}$ is finite. Since G is connected, for any morphism $f\colon b \to c$ in G, one may write $f = g^*h$ for some $g, h \in {}_aG$ so $\mathrm{mor}(G)$ is finite, proving (a). For (b), note that $\mathscr{R}$ cocycle finite implies that $\mathscr{R}$ is interval finite, so G is finite by (a). For $a \in \mathrm{ob}(G)$, ${}_a\Lambda = [0, e_a]_{{}_a\Lambda} = [0, N(\omega(a))]_{{}_a\Lambda}$ which is finite since $\mathscr{R}$ is cocycle finite. This proves (b). Finally, (c) follows from (b) since if $\mathscr{R}$ is principal, it is cocycle finite. $\square$

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